Constraints

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Differential Constraints

thanks to Adrew Witkin and Zoran Popivić

Differential Constraints

Beyond Points and Springs

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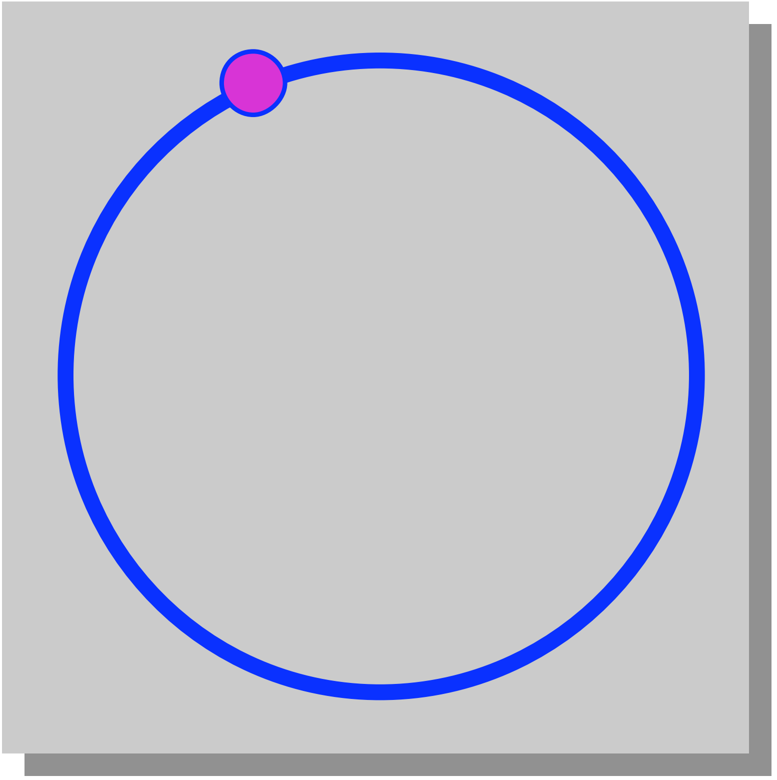
Beyond Points and Springs

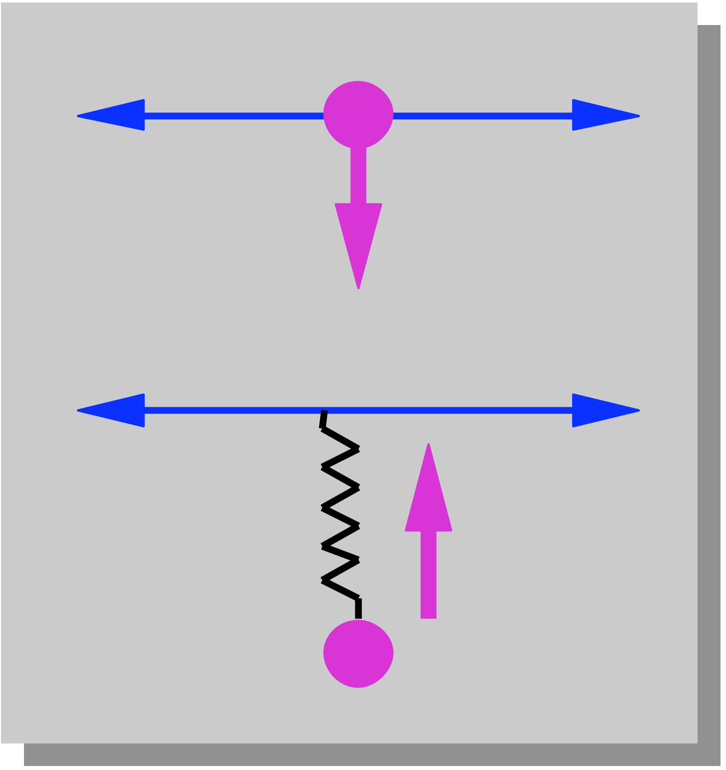
- You can make just about anything out of point masses and springs, *in principle*
- In practice, you can make anything you want as long as it's jello
- Constraints will buy us:
  - Rigid links instead of goopy springs
  - Ways to make interesting contraptions

A bead on a wire



- **Desired Behavior:**
  - **The bead can slide freely *along* the circle**
  - **It can never come off, however hard we pull**
- **Question:**
  - **How does the bead move under applied forces?**

Penalty Constraints

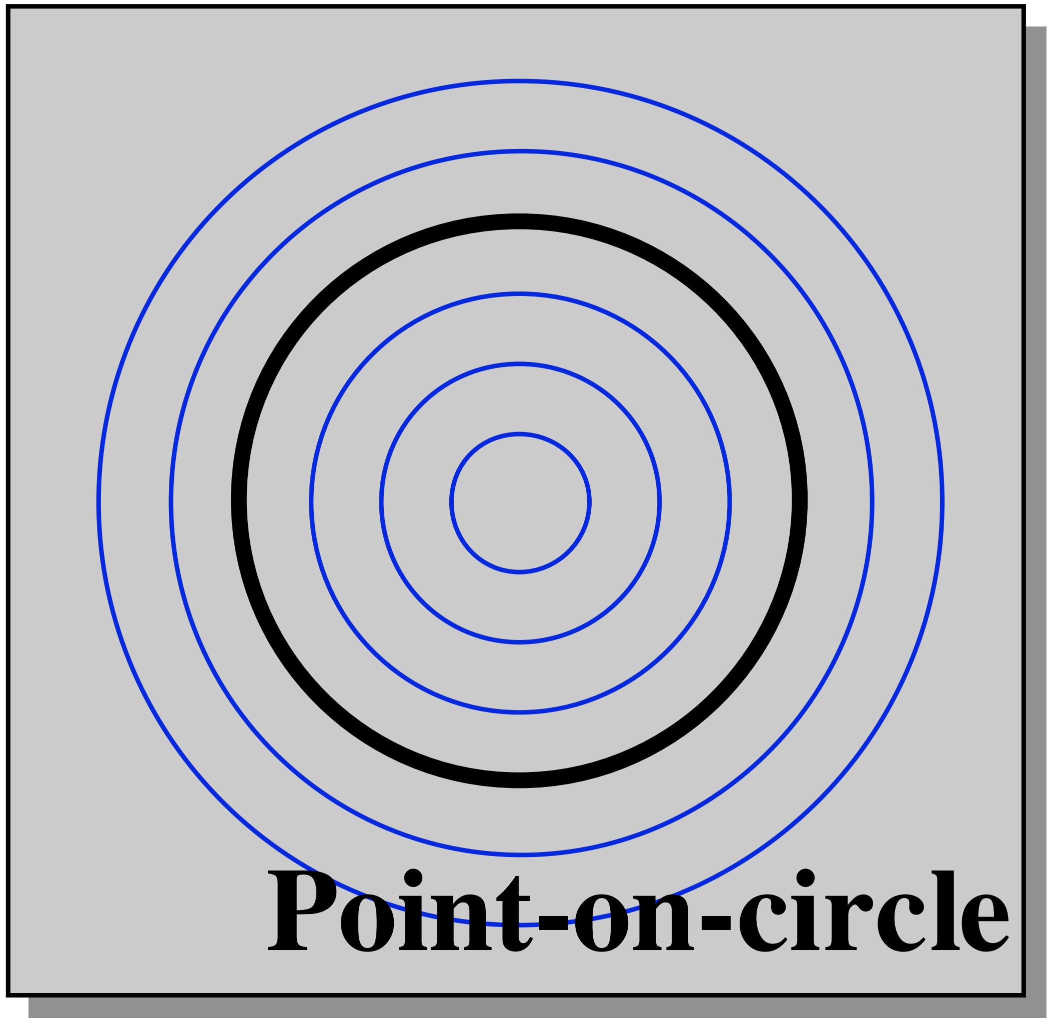


- **Why not use a spring to hold the bead on the wire?**
- **Problem:**
  - **Weak springs ⇒ goopy constraints**
  - **Strong springs ⇒ neptune express!**
- **A classic *stiff system***

Now for the Algebra …

- Fortunately, there's a general recipe for calculating the constraint force
- First, a single constrained particle
- Then, generalize to constrained particle systems

Representing Constraints



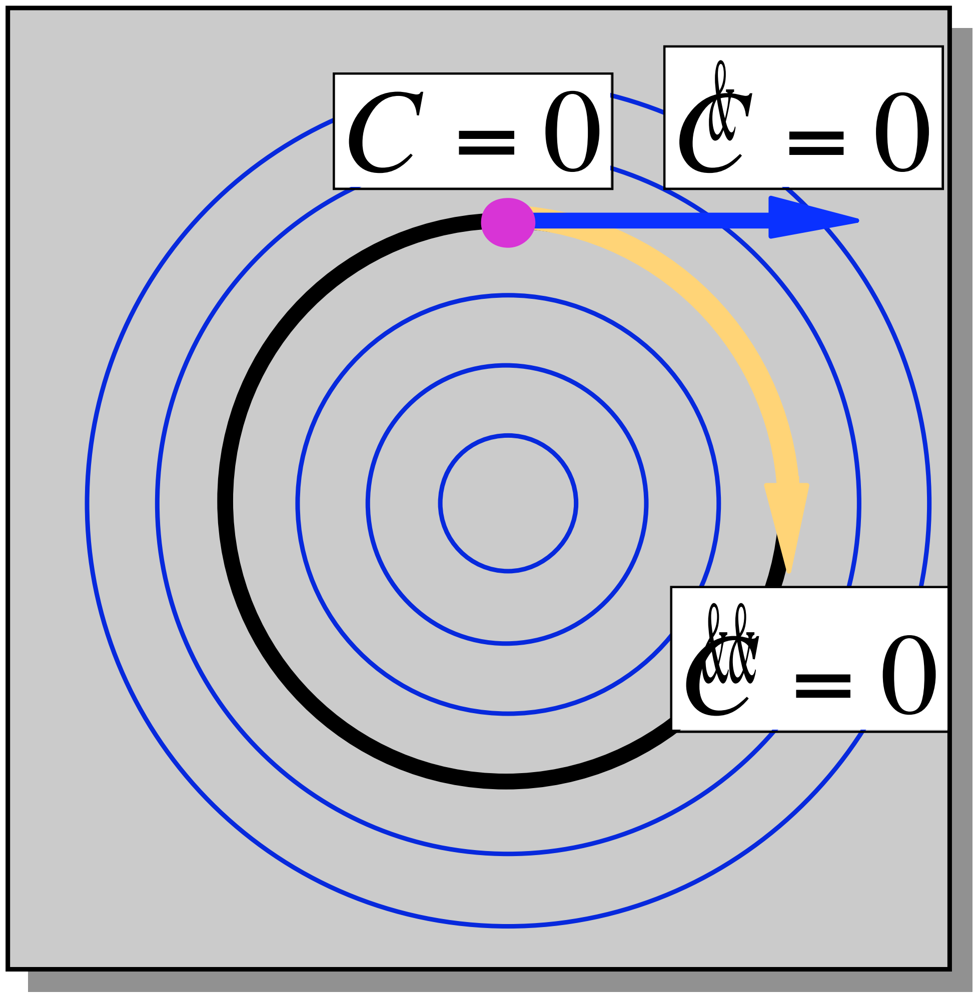
I. Implicit:

$$C(\mathbf{x}) = |\mathbf{x}| - r = 0$$

~~*II. Parametric:*~~

$$\cancel{\mathbf{x} = r\left[\cos\theta, \sin\theta\right]}$$

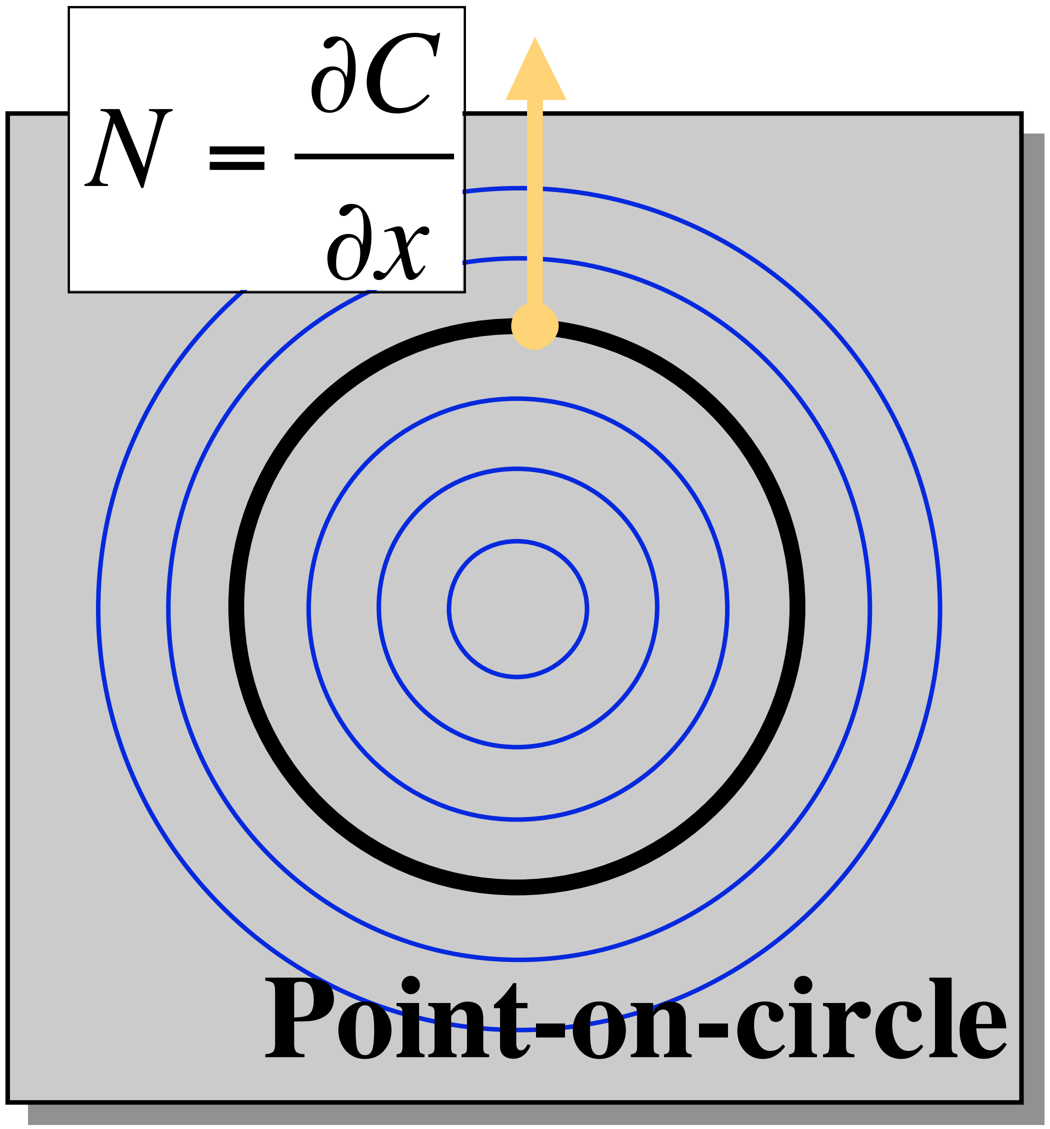
Maintaining Constraints Differentially



- **Start with legal position and velocity.**
- **Use constraint forces to ensure legal curvature.**

$C = 0$	legal position
$\dot{C} = 0$	legal velocity
$\ddot{C} = 0$	legal curvature

Constraint Gradient



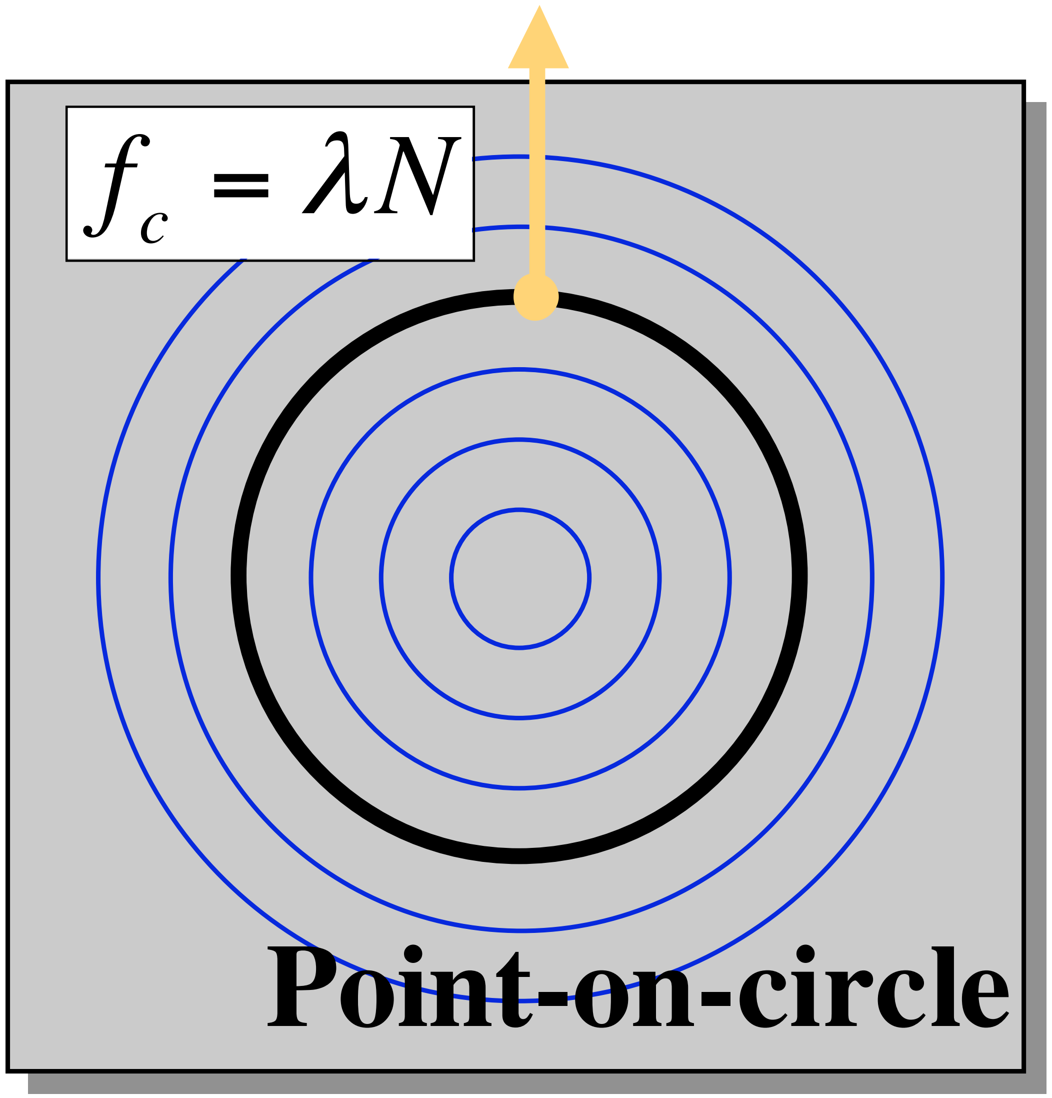
Implicit:

$$C(\mathbf{x}) = |\mathbf{x}| - r = 0$$

Differentiating C gives a normal vector.

This is the direction our constraint force will point in.

Constraint Forces



Constraint force: gradient vector times a scalar λ

Just one unknown to solve for

Assumption: constraint is passive—no energy gain or loss

Constraint Force Derivation

$$\boxed{C\left(x(t)\right)}$$

$$\dot{C} = N \cdot \dot{x}$$

$$\ddot{C} = \frac{\partial}{\partial t}\left(N \cdot x\right)$$

$$= \dot{N} \cdot \dot{x} + N \cdot \ddot{x}$$

$$\ddot{x} = \frac{f + f_c}{m}$$

$$f_c = \lambda N$$

Set $\ddot{C} = 0$, solve for λ:

$$\lambda = -m\frac{\dot{N} \cdot \dot{x}}{N \cdot N} - \frac{N \cdot f}{N \cdot N}$$

Constraint force is λN.

Notation: $N = \dfrac{\partial C}{\partial x}, \dot{N} = \dfrac{\partial^2 C}{\partial x \partial t}$

Example: Point-on-circle

$$C = |\mathbf{x}| - r$$

$$\mathbf{N} = \frac{\partial C}{\partial \mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|}$$

$$\dot{\mathbf{N}} = \frac{\partial^2 C}{\partial \mathbf{x} \partial t} = \frac{1}{|\mathbf{x}|}\left[\dot{\mathbf{x}} - \frac{\mathbf{x}\cdot\dot{\mathbf{x}}}{\mathbf{x}\cdot\mathbf{x}}\mathbf{x}\right]$$

Write down the constraint equation.

Take the derivatives.

Substitute into generic template, simplify.

$$\lambda = -m\frac{\dot{\mathbf{N}}\cdot\dot{\mathbf{x}}}{\mathbf{N}\cdot\mathbf{N}} - \frac{\mathbf{N}\cdot\mathbf{f}}{\mathbf{N}\cdot\mathbf{N}} = \left[m\frac{(\mathbf{x}\cdot\dot{\mathbf{x}})^2}{\mathbf{x}\cdot\mathbf{x}} - m(\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}) - \mathbf{x}\cdot\mathbf{f}\right]\frac{1}{|\mathbf{x}|}$$

Tinkertoys

- **Now we know how to simulate a bead on a wire.**
- **Next: a constrained particle *system*.**
  - **E.g. constrain particle/particle distance to make rigid links.**
- **Same idea, but…**

Compact Particle System Notation

$$\ddot{\mathbf{q}} = \mathbf{W}\mathbf{Q}$$

q: *3n*-long *state vector.*

Q: *3n*-long *force vector.*

M: *3n* x *3n* diagonal *mass matrix*.

W: M-inverse (element- wise reciprocal)

$$\mathbf{q} = [\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n]$$

$$\mathbf{Q} = [\mathbf{f}_1, \mathbf{f}_2, \cdots, \mathbf{f}_n]$$

$$\mathbf{M} = \begin{bmatrix} m_1 & & & & & & \\ & m_1 & & & & & \\ & & m_1 & & & & \\ & & & \ddots & & & \\ & & & & m_n & & \\ & & & & & m_n & \\ & & & & & & m_n \end{bmatrix}$$

$$\mathbf{W} = \mathbf{M}^{-1}$$

$$\mathbf{C} = [C_1, C_2, \cdots, C_m]$$

$$\lambda = [\lambda_1, \lambda_2, \cdots, \lambda_m]$$

$$\mathbf{J} = \frac{\partial \mathbf{C}}{\partial \mathbf{q}}$$

$$\dot{\mathbf{J}} = \frac{\partial^2 \mathbf{C}}{\partial \mathbf{q} \partial t}$$

Solving for the Constraint Force

$$\ddot{x} = \frac{1}{m}(f + \hat{f})$$

$$C(x) = x \cdot x - 1 = 0$$

$$\dot{C}(x) = 2x \cdot \dot{x} = 0$$

$$\ddot{C}(x) = 2(x \cdot \ddot{x} + \dot{x} \cdot \dot{x})$$

$$\dot{x} \cdot \dot{x} + x \cdot \left(\frac{1}{m}(f + \hat{f})\right) = 0$$

$$x \cdot \hat{f} = -m\dot{x} \cdot \dot{x} - x \cdot f$$

Bead on a Wire

$$\ddot{q} = W(Q + \hat{Q})$$

$$C(q) = 0$$

$$\dot{C} = \frac{\partial C}{\partial q}\dot{q} = J\dot{q}$$

$$\ddot{C} = \dot{J}\dot{q} + J\ddot{q}$$

$$\dot{J}\dot{q} + JW(Q + \hat{Q}) = 0$$

$$JW\hat{Q} = -\dot{J}\dot{q} - JWQ$$

General Case

Force Must be a Linear Combination of Constraint Graidntes

Bead on a Wire

$$T = \frac{1}{2} m \dot{x} \cdot \dot{x}$$

$$\dot{T} = m \dot{x} \cdot \ddot{x} = 0$$

$$\ddot{x} \leftarrow \frac{1}{m} \hat{f}$$

$$\dot{x} \cdot \hat{f} = 0$$

$$\hat{f} = \lambda \vec{x}$$

General Case

$$T = \frac{1}{2} \dot{q}^T M \dot{q}$$

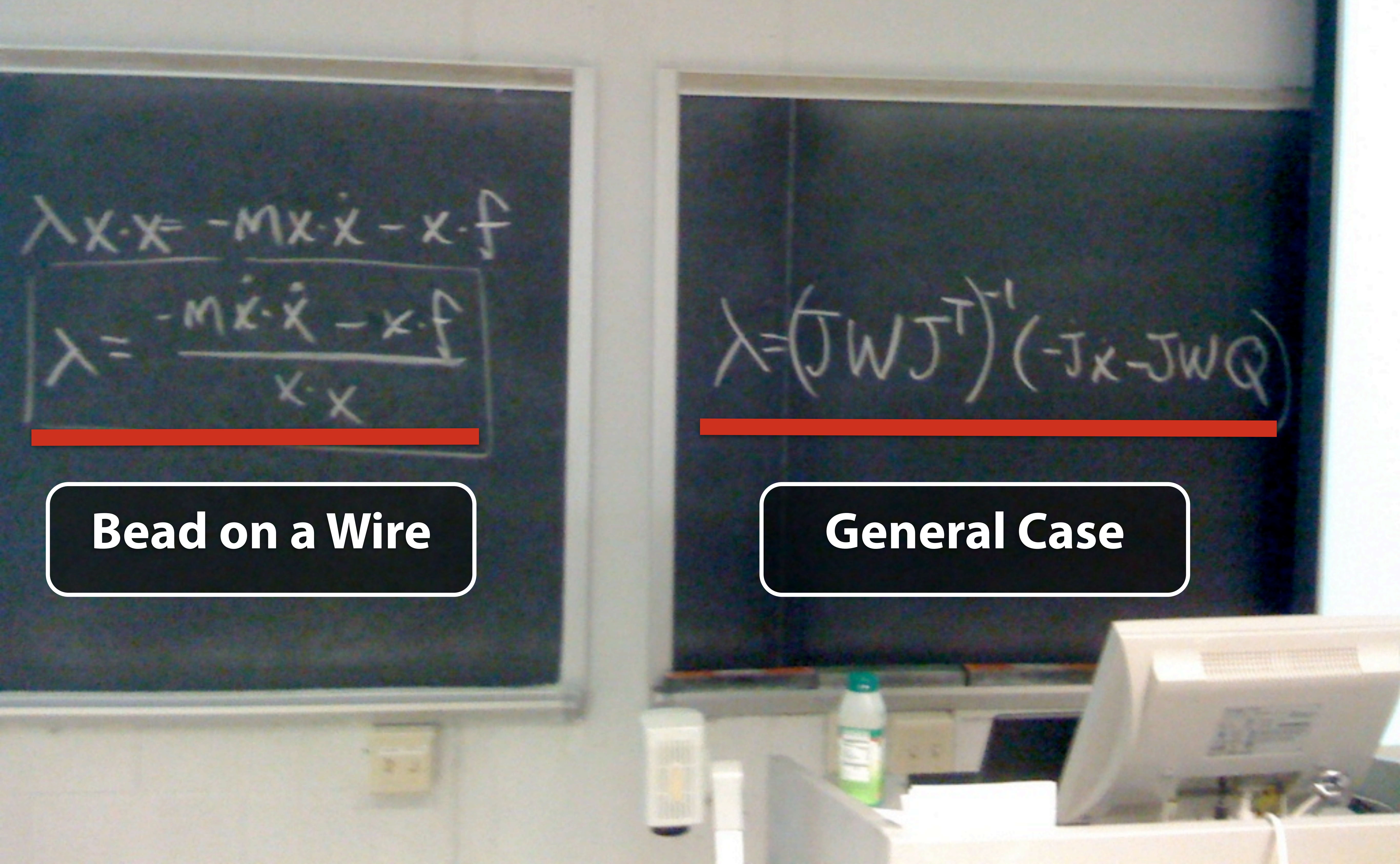
$$\dot{T} = \dot{q}^T M \ddot{q} = 0$$

$$= \dot{q}^T M W \hat{Q}$$

$$= \dot{q}^T \hat{Q} = 0$$

$$\Rightarrow \hat{Q} = J^T \lambda$$

Final Solution for the λ Multipliers



Particle System Constraint Equations

Matrix equation for λ

$$\left[\mathbf{J}\mathbf{W}\mathbf{J}^T\right]\lambda = -\dot{\mathbf{J}}\dot{\mathbf{q}} - \left[\mathbf{J}\mathbf{W}\right]\mathbf{Q}$$

Constrained Acceleration

$$\ddot{\mathbf{q}} = \mathbf{W}\left[\mathbf{Q} + \mathbf{J}^T\lambda\right]$$

More Notation

$$\mathbf{C} = \left[C_1, C_2, \cdots, C_m\right]$$

$$\lambda = \left[\lambda_1, \lambda_2, \cdots, \lambda_m\right]$$

$$\mathbf{J} = \frac{\partial \mathbf{C}}{\partial \mathbf{q}}$$

$$\dot{\mathbf{J}} = \frac{\partial^2 \mathbf{C}}{\partial \mathbf{q}\, \partial t}$$

Derivation: just like bead-on-wire.

Drift and Feedback

- **In principle, clamping $\ddot{C}$ at zero is enough**
- **Two problems:**
  - **Constraints might not be met initially**
  - **Numerical errors can accumulate**
- **A feedback term handles both problems:**

$$\ddot{C} = -\alpha C - \beta\dot{C}, \text{ instead of}$$

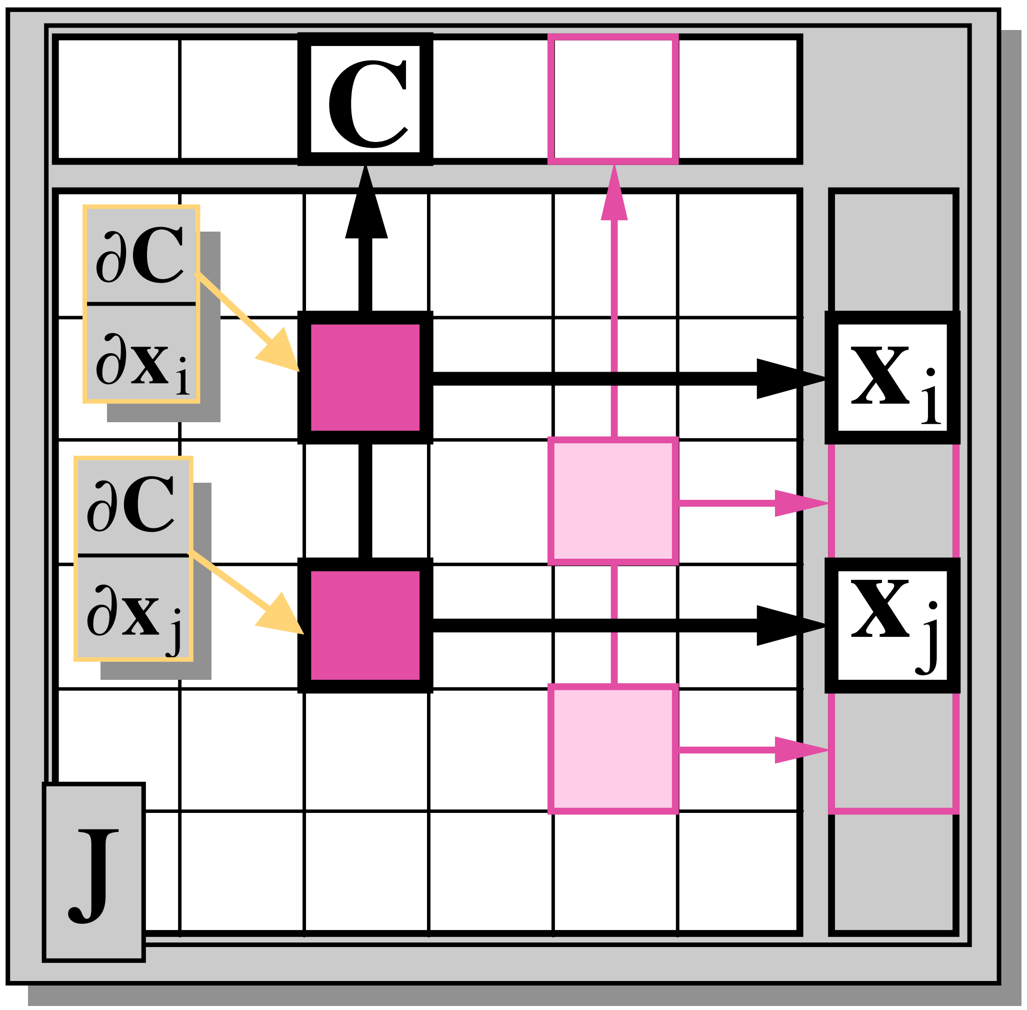
$$\ddot{C} = 0$$

$\boxed{\alpha \text{ and } \beta \text{ are magic constants.}}$

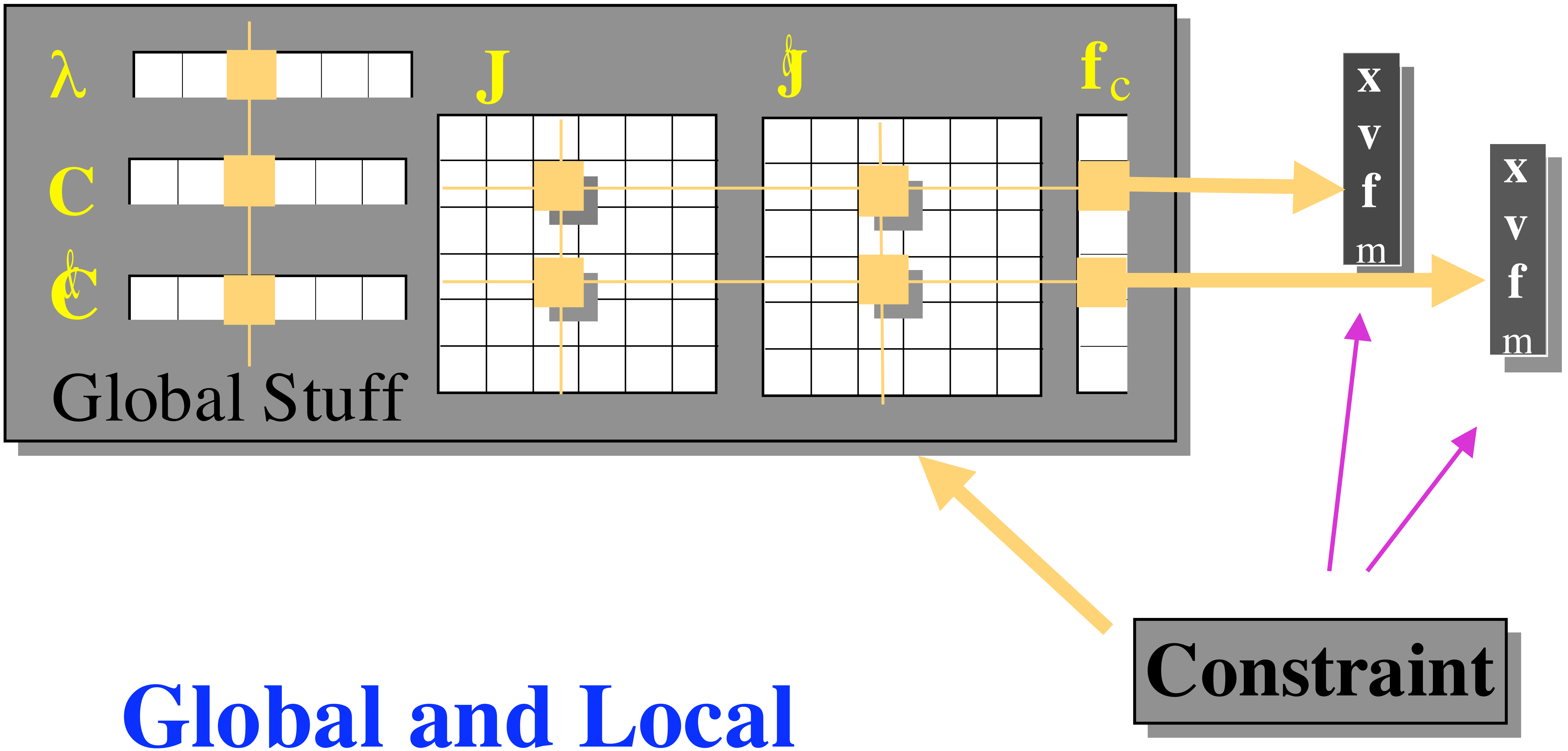
How do you implement all this?

- We have a global matrix equation.
- We want to build models on the fly, just like masses and springs.
- Approach:
  - Each constraint adds its own piece to the equation.

Matrix Block Structure



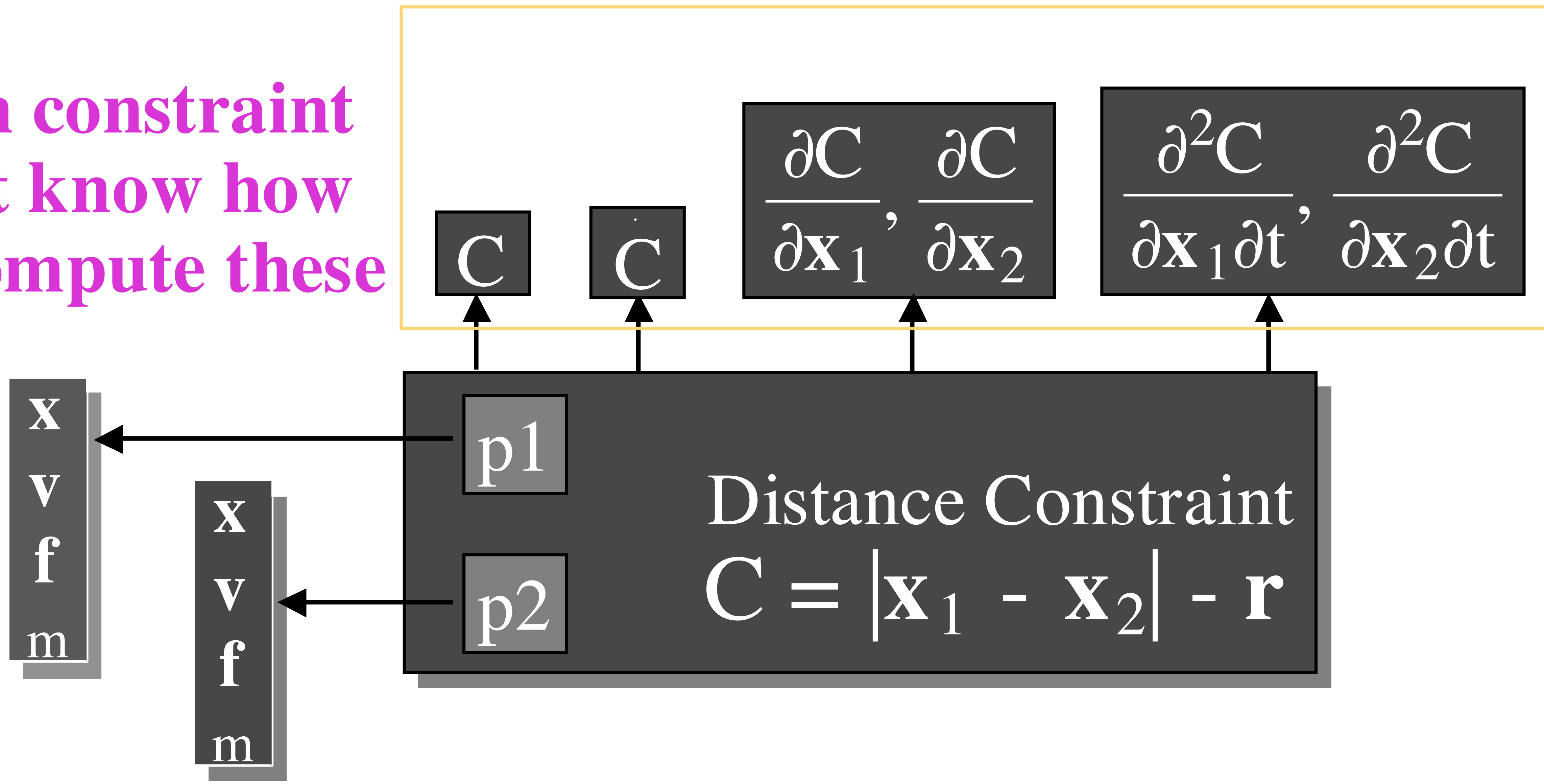
- Each constraint contributes one or more *blocks* to the matrix.
- Sparsity: many empty blocks.
- Modularity: let each constraint compute its own blocks.
- Constraint and particle indices determine block locations.



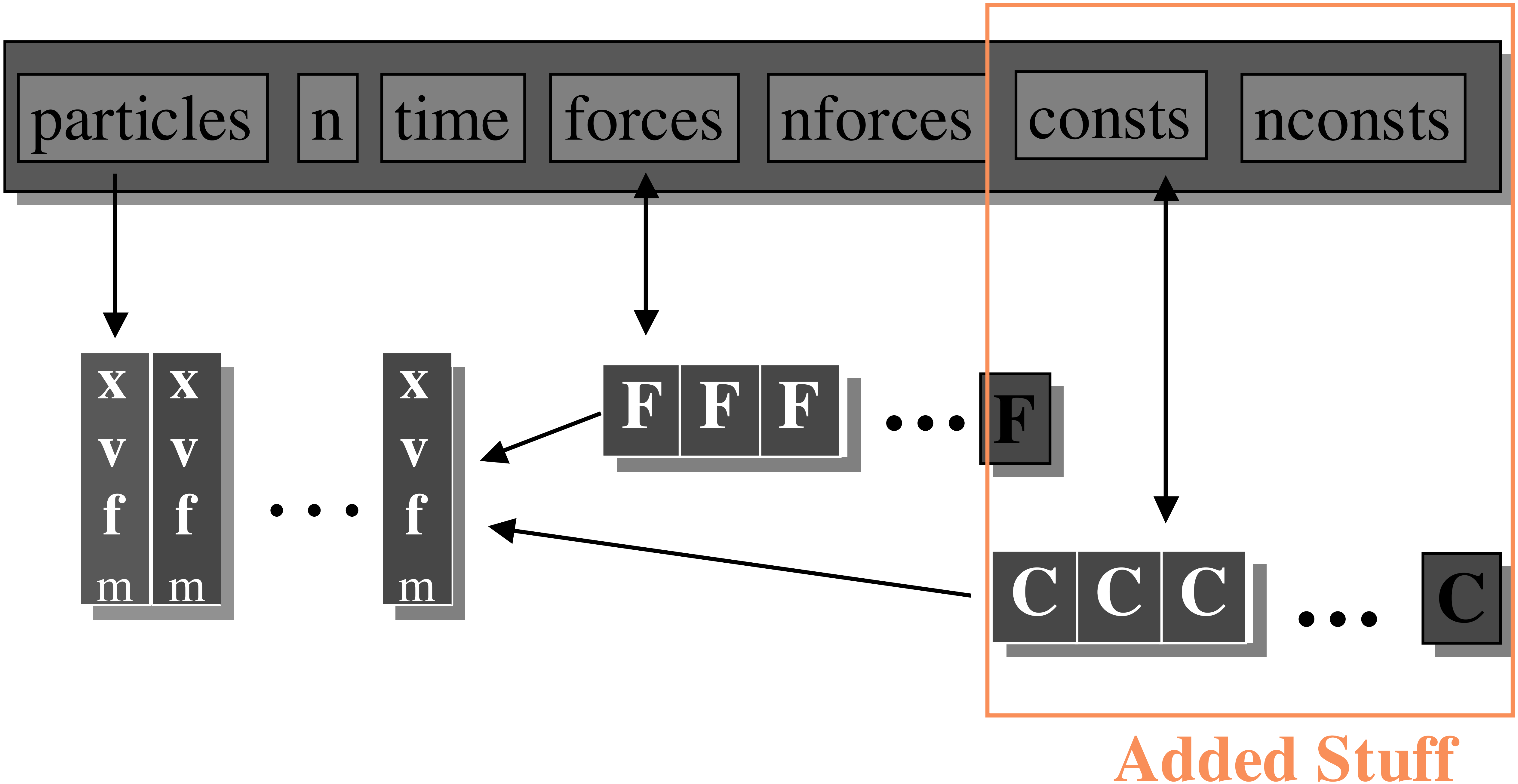
Global and Local

Constraint Structure

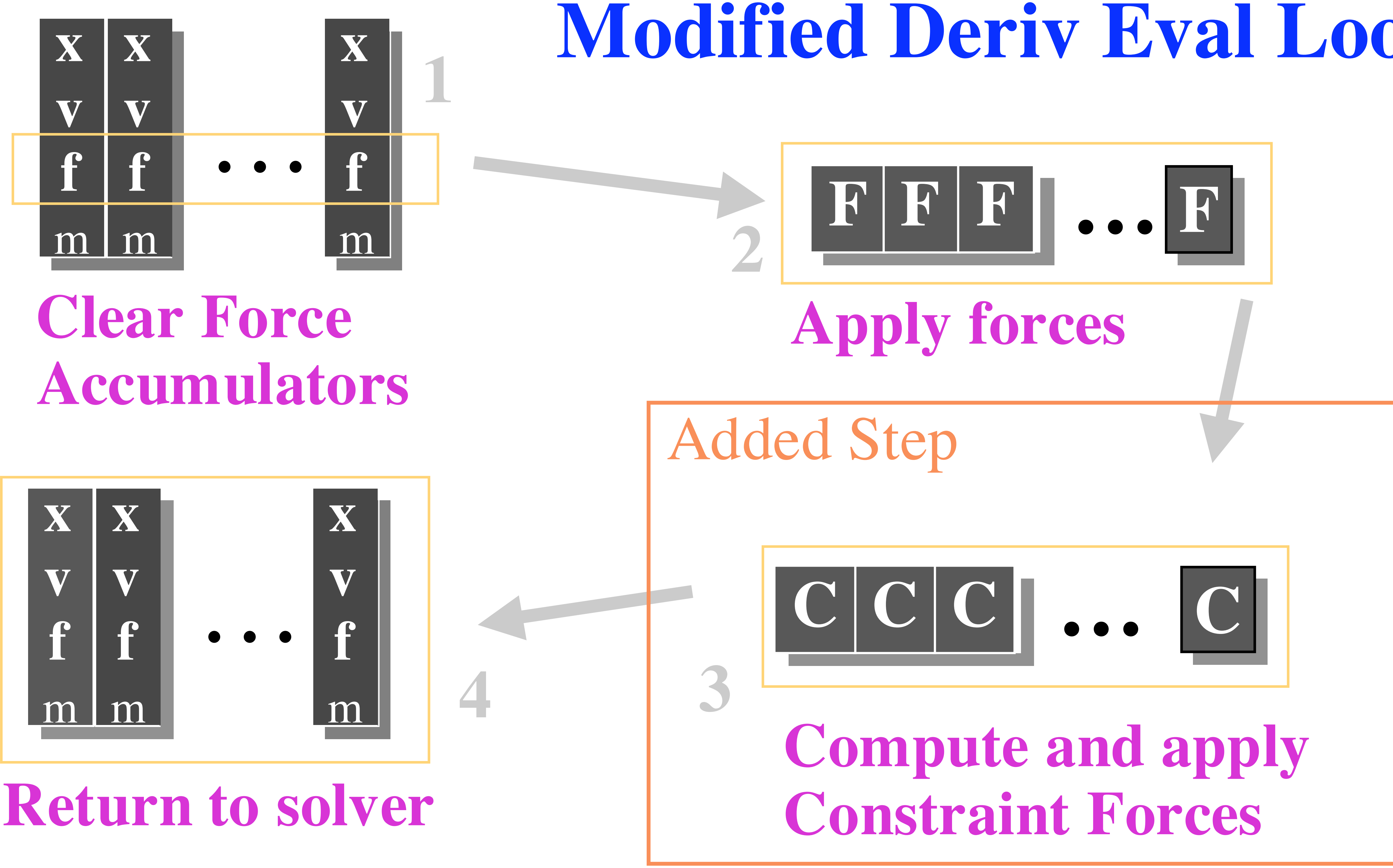
Each constraint must know how to compute these



Constrained Particle Systems



Modified Deriv Eval Loop



Constraint Force Eval

- **After computing ordinary forces:**
  - **Loop over constraints, assemble global matrices and vectors.**
  - **Call matrix solver to get λ, multiply by J^T to get constraint force.**
  - **Add constraint force to particle force accumulators.**

Impress your Friends

- **The requirement that constraints not add or remove energy is called the *Principle of Virtual Work*.**
- **The λ's are called *Lagrange Multipliers*.**
- **The derivative matrix, J, is called the *Jacobian Matrix*.**