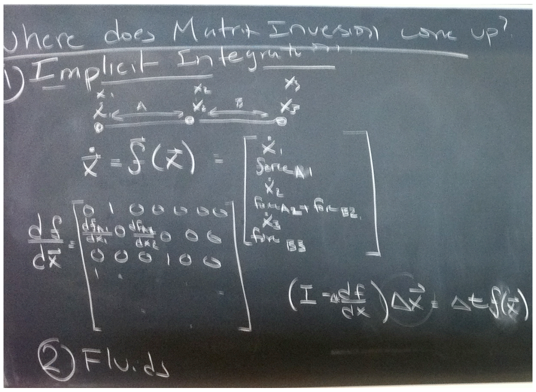
The Conjugate Gradient Method

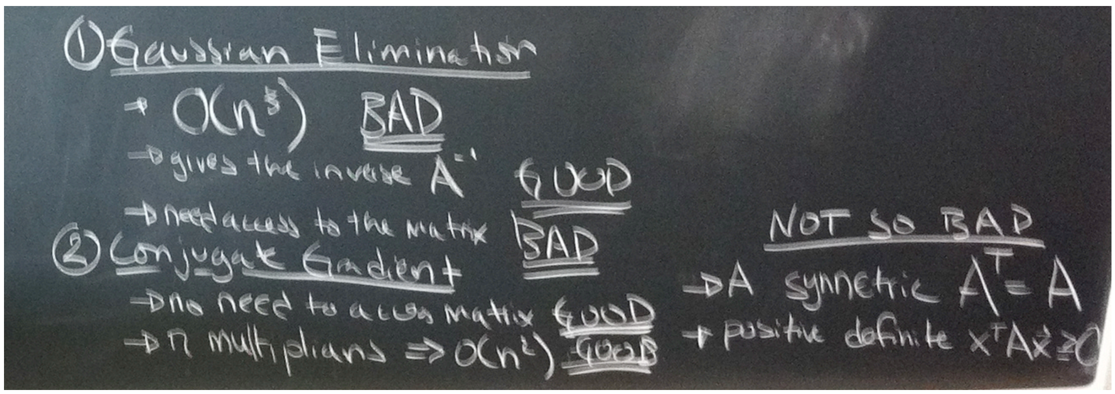
15-867: The Animation of Natural Phenomena

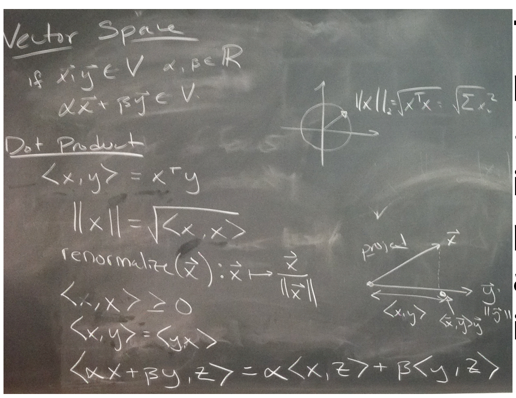


We often see linear systems ($Ax=b$), which are
- sparse,
- symmetric, and
- positive definite.

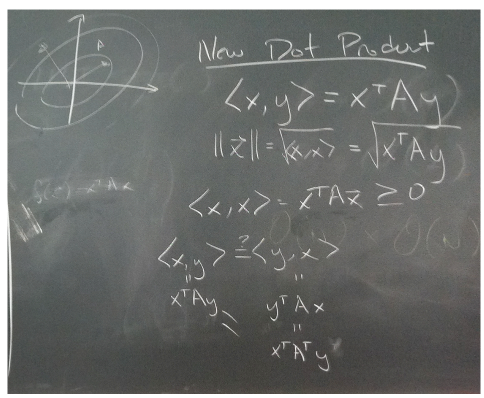
An example is the linearized implicit update step.

We could solved this with Guassian Elimination, for example, but a much better idea is the Conjugate Gradient method, which only requires matrix multiplies.





The ordinary dot product $\langle x,y\rangle = x^T y$ is symmetric, positive definite, and linear in both it's arguments.



...and so is this generalized dot product: $\langle x,y\rangle_A = x^T A y$ (You can check!)

Using this generalized dot product ($\langle x,y\rangle_A$), we can create a basis P which is orthonormal under the dot product:

(1) $P^T A P = I$

but, recall that we wanted to solve the following equation:

(2) $Ax = b$

the first step is to multiply (2) by P^T

(3) $P^T A x = P^T b$

the create a new variable y such that

(4) $x = Py$

Substituting (1) and (4) into (3) we get:

(5) $P^T A P y = y = P^T b$

which allows us to solve for x

(6) $x = Py = PP^T b$

The key to the conjugate gradient method is to build up a sequence of bases $P_1, P_2, \ldots$ one vector at a time (orthonormalizing each new vector to ensure $P_i^T A P_i = I$), and solve for successive approximations to $x_1, x_2, \ldots \to x$ by using equation (6): $x_i = P_i P_i^T b$