

Implicit Integration + Large Linear Systems

15-867: The Animation of Natural Phenomena

Implicit Integration

$$\vec{x}(t+\Delta t) = \vec{x}(t) + \Delta t \vec{f}(\vec{x}(t+\Delta t))$$

$$\vec{x}(t) = \vec{f}(\vec{x})$$

$$\vec{f}(\vec{x}) \approx \vec{f}(\vec{x}_0) + \frac{d\vec{f}}{d\vec{x}}(\vec{x} - \vec{x}_0)$$

$$\vec{x}(t+\Delta t) - \vec{x}(t) \approx \Delta \vec{x} = \Delta t \left(\vec{f}(\vec{x}(t)) + \frac{d\vec{f}(\vec{x}(t))}{d\vec{x}} \Delta \vec{x} \right)$$

$$\left(\mathbf{I} - \Delta t \frac{d\vec{f}}{d\vec{x}} \right) \Delta \vec{x} = \Delta t \vec{f}(\vec{x})$$

We can approximate implicit integration of a nonlinear system as follows. The integrator turns into a solution to a linear system. The key is the large sparse **Jacobian** matrix, df/dx .

$\frac{d\vec{f}}{d\vec{x}}$ sparse symmetric positive definite 12×12 matrix

12 particles
72 variables
 $\frac{d\vec{f}}{d\vec{x}} \in \mathbb{R}^{72 \times 72}$

vec Conjugate-gradient ($A, \vec{x}_0, \vec{b}$)

// solution $A\vec{x} = \vec{b}$

double $\vec{x} \rightarrow A$

vec $A(\text{vec } \vec{x})$; // function that "acts like A"

system

getState
setState
getDeriv
multByJacobian ($\frac{d\vec{f}}{d\vec{x}}$)

The Jacobian is large, sparse, symmetric and positive definite. This allows us to solve this system with the **conjugate gradient** method.

Implicit Integration Code

$$\left(\mathbf{I} - \Delta t \frac{d\vec{f}}{d\vec{x}} \right) \Delta \vec{x} = \Delta t \vec{f}(\vec{x})$$

```

void Implicit(sys, Δt) {
    f ← sys.getDeriv(),
    b ← Δt * f
    vec lhs(Δx) { // return (I - Δt * df/dx) Δx
        Δx' = Δt * sys.multByJacobian(Δx)
        return Δx - Δx'
    }
    Δx ← conjugateGradient(lhs, 0, b)
    sys.setState(sys.getState() + Δx)
}
    
```

getState
setState
getDeriv
multByJacobian ($\frac{d\vec{f}}{d\vec{x}}$)
conjugateGradient ($A, \vec{x}_0, \vec{b}$)

Using the conjugate gradient method, we can now write down **pseudocode for implicit integration**. We only need:

- getState
- setState
- getDeriv
- multByJacobian
- conjugateGradient

Useful notes:

$$\frac{d}{dx} |x| = \frac{x}{|x|} \quad \left\{ \begin{array}{l} \text{when } x \neq 0 \\ \frac{d}{dx} \frac{x}{|x|} = 0 \end{array} \right.$$

Force = $-f_{10}$

Force = $(k_2 x_1 - \tau) \left(\frac{x_1 - x_2}{k_2 x_1} \right)$

Force = $-f_{20}$

Question: What is the Jacobian for three "beads on a wire" connected by undamped linear springs?