DiffEQ 1

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A Canonical Differential Equation

\[ \dot{x} = f(x, t) \]

- \( x(t) \): a moving point.
- \( f(x, t) \): \( x \)'s velocity.
The differential equation
\[ \dot{x} = f(x, t) \]
defines a vector field over \( x \).
Integral Curves

Start Here

Pick any starting point, and follow the vectors.
Given the starting point, follow the integral curve.
Euler’s Method

- Simplest numerical solution method
- Discrete time steps
- Bigger steps, bigger errors.

\[ x(t + \Delta t) = x(t) + \Delta t f(x, t) \]
Two Problems

- Accuracy
- Instability
Consider the equation:

\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x
\]

What do the integral curves look like?
Problem I: Inaccuracy

Error turns $x(t)$ from a circle into the spiral of your choice.
Problem 2: Instability

- Consider the following system:

\[
\begin{align*}
\dot{x} &= -x \\
x(0) &= 1
\end{align*}
\]
Problem 2: Instability

Euler's Method

- Simplest numerical solution method
- Discrete time steps
- Bigger steps, bigger errors.

Problem I: Inaccuracy

Error turns $x(t)$ from a circle into the spiral of your choice.

Problem II: Instability to Neptune!
Accuracy of Euler Method

\[ \dot{x} = f(x) \]
Accuracy of Euler Method

\[ \dot{x} = f(x) \]

Consider Taylor Expansion about \( x(t) \)...
Accuracy of Euler Method

\[ \dot{x} = f(x) \]

Consider Taylor Expansion about \( x(t) \)...
Accuracy of Euler Method

$$\dot{x} = f(x)$$

Consider Taylor Expansion about $x(t)$...

$$x(t + h) = \text{constant}$$
Accuracy of Euler Method

\[ \dot{x} = f(x) \]

Consider Taylor Expansion about \( x(t) \)...

\[ x(t + h) = x(t) + \ldots \]

constant
Accuracy of Euler Method

\[ \dot{x} = f(x) \]

Consider Taylor Expansion about \( x(t) \)...

\[ x(t + h) = x(t) + \ldots \]

constant  linear
Accuracy of Euler Method

\[ \dot{x} = f(x) \]

Consider Taylor Expansion about \( x(t) \)... 

\[ x(t + h) = x(t) + hf(x(t)) + \ldots \]

constant \hspace{1cm} linear
Accuracy of Euler Method

\[ \dot{x} = f(x) \]

Consider Taylor Expansion about \( x(t) \)...

\[ x(t + h) = x(t) + hf(x(t)) + \ldots \]

constant, linear, everything else
Accuracy of Euler Method

\[ \dot{x} = f(x) \]

Consider Taylor Expansion about \( x(t) \)...

\[ x(t + h) = x(t) + hf(x(t)) + O(h^2) \]

constant \hspace{1cm} linear \hspace{1cm} everything else
Accuracy of Euler Method

\[ \dot{x} = f(x) \]

Consider Taylor Expansion about \( x(t) \):

\[ x(t + h) = x(t) + hf(x(t)) + O(h^2) \]

Euler step
- constant
- linear
- everything else
Accuracy of Euler Method

\[ \dot{x} = f(x) \]

Consider Taylor Expansion about \( x(t) \)...

\[ x(t + h) = x(t) + h f(x(t)) + O(h^2) \]

- \( x(t + h) \): Euler step
- \( x(t) \): constant
- \( h f(x(t)) \): linear
- \( O(h^2) \): everything else
Accuracy of Euler Method

\[ \dot{x} = f(x) \]

Consider Taylor Expansion about \( x(t) \)...

\[ x(t + h) = x(t) + hf(x(t)) + O(h^2) \]

Euler step

\[ \begin{align*}
\text{constant} & \quad \text{linear} \\
O(h^2) & \quad \text{everything else}
\end{align*} \]

Therefore, Euler’s method has error \( O(h^2) \)... it is first order.
Accuracy of Euler Method

\[ \dot{x} = f(x) \]

Consider Taylor Expansion about \( x(t) \)...

\[
x(t + h) = x(t) + hf(x(t)) + O(h^2)
\]

Therefore, Euler’s method has error \( O(h^2) \)... it is first order.

How can we get to \( O(h^3) \) error?
The Midpoint Method

- Also known as second order Runge-Kutte:

\[
k_1 = h(f(x_0, t_0))
\]

\[
k_2 = hf(x_0 + \frac{k_1}{2}, t_0 + \frac{h}{2})
\]

\[
x(t_0 + h) = x_0 + k_2 + O(h^3)
\]
The Midpoint Method

a. Compute an Euler step
\[ \Delta x = \Delta t f(x, t) \]

b. Evaluate \( f \) at the midpoint
\[ f_{\text{mid}} = f \left( \frac{x + \Delta x}{2}, \frac{t + \Delta t}{2} \right) \]

c. Take a step using the midpoint value
\[ x(t + \Delta t) = x(t) + \Delta t f_{\text{mid}} \]
q-Stage Runge-Kutta

General Form:

\[ x(t_0 + h) = x_0 + h \sum_{i=1}^{q} w_i k_i \]

where:

\[ k_i = f \left( x_0 + h \sum_{j=1}^{i-1} \beta_{ij} k_j \right) \]

Find the constant that ensure accuracy O(h^n).
4th-Order Runge-Kutta

\[ k_1 = hf(x_0, t_0) \]

\[ k_2 = hf(x_0 + \frac{k_1}{2}, t_0 + \frac{h}{2}) \]

\[ k_3 = hf(x_0 + \frac{k_2}{2}, t_0 + \frac{h}{2}) \]

\[ k_4 = hf(x_0 + k_3, t_0 + h) \]

\[ x(t_0 + h) = x_0 + \frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4 + O(h^5) \]
4th-Order Runge-Kutta

\[ k_1 = hf(x_0, t_0) \]

\[ k_2 = hf(x_0 + \frac{k_1}{2}, t_0 + \frac{h}{2}) \]

\[ k_3 = hf(x_0 + \frac{k_2}{2}, t_0 + \frac{h}{2}) \]

\[ k_4 = hf(x_0 + k_3, t_0 + h) \]

\[ x(t_0 + h) = x_0 + \frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4 + O(h^5) \]

Why so popular?
<table>
<thead>
<tr>
<th>Order</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stages</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>7</td>
<td>9</td>
<td>11</td>
</tr>
</tbody>
</table>
More methods...

- Euler’s method is *1st Order*.
- The midpoint method is *2nd Order*.
- Just the tip of the iceberg. See *Numerical Recipes* for more.

- **Helpful hints:**
  - *Don’t* use Euler’s method (you will anyway.)
  - *Do* use adaptive step size.
Modular Implementation

• Generic operations:
  – Get dim(x)
  – Get/set x and t
  – Deriv Eval at current (x,t)

• Write solvers in terms of these.
  – Re-usable solver code.
  – Simplifies model implementation.
Solver Interface

System

Solver

Dim(state)

Get/Set State

Deriv Eval
void eulerStep(Sys sys, float h) {
    float t = getTime(sys);
    vector<float> x0, deltaX;

    t = getTime(sys);
    x0 = getState(sys);
    deltaX = derivEval(sys, x0, t);
    setState(sys, x0 + h*deltaX, t+h);
}
What sorts of common physical phenomena are not well modeled by differential equations?
What sorts of phenomena are not well modeled?

electrons - probabilistic **WRONG**
theory of relativity **WRONG**
sudden forces - not continuous
interactions between multiple particles **WRONG**
anything nondeterministic **WRONG**
shattering