# Lecture Notes: Resistive Model of a Graph and Random Walks 

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## 0 Motivation

Consider a recommendation system that recommends viewers to movie titles. We can represent this system as a bipartite graph $G=(L, R, E)$ with a ranking function $r$ where $L$ is the set of viewers, $R$ is the set of movies, $E$ is the set of edges, and $r: E \rightarrow \mathbb{N}$ gives for $(v, m) \in E$ viewer $v$ 's ranking of movie $m$.


Using this model, we want to answer the following question: Should we recommend movie $m$ to viewer $v$ ? In other words, how can we assign some sort of score $\operatorname{score}(v, m)$ to a recommendation of $m$ to $v$. (Note: $(v, m)$ need not be an edge in $G$ - we want to be able to generate these recommendation scores regardless of whether a $v$ has a ranking for $m$ ). Let's explore a couple of ideas.

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## Idea 1

$$
\operatorname{score}(v, m)=\frac{1}{\operatorname{dist}_{G}(v, m)}
$$

where we define $\operatorname{dist}_{G}(v, m)$ as follows: Assign a weight to each edge $(i, j)$ in the graph given by

$$
w_{i j}=\frac{1}{r(i, j)}
$$

and the weight of a path $P$ is given by

$$
w(P)=\sum_{e \in P} w_{e}
$$

Now, let

$$
\operatorname{dist}_{G}(v, m)=\min _{v P m} W(P)
$$

## Idea 2

$$
\operatorname{score}(v, m)=\max _{v P m} w(P)
$$

where we define

$$
w(P)=\min _{e \in P} r(e)
$$

However, these two ideas don't help us since we never take into account multiple paths between $v$ and $m$ to add to the score. Intuitively, the more paths there are between a viewer and a movie, the higher the score $(v, m)$ should be. We can amend this by considering the following idea:

## Idea 3

$$
\operatorname{score}(v, m)=\max \text { flow from } v \text { to } m
$$

This is still not good enough, since max flow won't reward shorter paths between viewers and movies. This motivates two other ideas:

## Idea 4

View the edges as conductors, and let

$$
\operatorname{score}(v, m)=\operatorname{hit}(v, m)+\operatorname{hit}(m, v)
$$

where hit $(v, m)$ denotes the expected length of a random walk from $v$ to $m$

## Idea 5

Consider a random walk from $v$ to $m$

$$
\operatorname{score}(v, m)=\text { "the effective conductance" between } v \text { and } m
$$

We will show that Ideas 4 and 5 are equal up to scaling. We will also explore whether effective conductance and "commute time" provide a better scoring system.

The rest of the lecture will proceed as follows:

- Provide formal definitions
- Develop a Basic Theory
- Give efficient algorithms
- Find applications


## 1 Resistance Theory

### 1.1 Some preliminaries

We'll first use some basic laws of physics adn shit

## Ohm's Law

Let $C=$ conductance, $R=$ resistance, $V=$ voltage, and $i=$ current, then

$$
C=\frac{1}{R} \quad i=C V=\frac{V}{R}
$$



## Resistors in Series

Series of resistors will act like a single resistor


## Conductors in Parallel

Conductors in parallel will act like a single conductor


$$
C=C_{1}+\ldots+C_{m}
$$

### 1.2 Effective Resistance


$v_{a b}$

Definition 1. Let $G$ be a network of resistors, where the conductance on each individual edge is given by the edge weight. The Effective Resistance (Conductance) between two vertices a and $b$ of in $G$ is the amount of electrical resistance (conductance) between them:

$$
R_{a b}=\frac{V_{a b}}{i_{a b}} \quad C_{a b}=\frac{1}{R_{a b}}
$$

Now, to compute the effective resistance from $a$ to $b$, we will make use of Kirchoff's Law

## Kirchoff's (1st) Law

The current flowing into a node must be equal to the current flowing out of it.
(This will be the case for all nodes other than $a$ and $b$ )

Now, consider the following example (the left is the weighted graph, while the right represents the voltage and current flow)


Using Ohm's Law, we have that

$$
\begin{aligned}
i_{1} & =C_{1}\left(V-V_{1}\right) \\
i_{2} & =C_{2}\left(V-V_{2}\right) \\
i_{3} & =C_{3}\left(V-V_{3}\right)
\end{aligned}
$$

The residual current is given by $i_{1}+i_{2}+i_{3}$. Kirchoff's Law tells us that this quantity must equal 0 , hence

$$
C_{1}\left(V-V_{1}\right)+C_{2}\left(V-V_{2}\right)+C_{3}\left(V-V_{3}\right)=0 \Longrightarrow\left(C_{1}+C_{2}+C_{3}\right) V=C_{1} V_{1}+C_{2} V_{2}+C_{3} V_{3}
$$

Setting $C=C_{1}+C_{2}+C_{3}$ gives us

$$
C V=C_{1} V_{1}+C_{2} V_{2}+C_{3} V_{3} \Longrightarrow V=\frac{C_{1}}{C} V_{1}+\frac{C_{2}}{C} V_{2}+\frac{C_{3}}{C} V_{3}
$$

Observe that we've written $V$ as a convex combination of $V_{1}, V_{2}, V_{3}$. The residual current is equal to $C V-C_{1} V_{1}-C_{2} V_{2}-C_{3} V_{3}$

Now, we'll consider the general case for some arbitrary network $G$, where $G=(V, E, C)$ where $C: E \rightarrow \mathbb{R}^{+}$and $V=\{1, \ldots, n\}$. Also, define

$$
d(a)=\sum_{(a, b) \in E} C(a, b)
$$

Define a corresponding adjacency matrix $A$ whose entries are given by

$$
A_{a b}= \begin{cases}C(a, b) & (a, b) \in E \\ 0 & o . w .\end{cases}
$$

The Laplacian matrix of $G$ (denoted by $L(G)$ or simply $L$ ) is given by

$$
L_{a b}= \begin{cases}d(a) & a=b \\ -C_{a b} & (a, b) \in E \\ 0 & \text { o.w. }\end{cases}
$$

Notice that

$$
L=D-A \quad \text { where } D=\left[\begin{array}{ccc}
d(1) & & 0 \\
& \ddots & \\
0 & & d(n)
\end{array}\right]
$$

Let $\mathbf{v}$ be a vector representing the voltage setting of each node; This would mean that $(L \mathbf{v})_{i}$ calculates the residual current at node $i$. Now, suppose we are interested in the inverse, i.e. we inject current into each node and observe the voltage. The net current injected should be zero (in order to abide by Kirchoff's law). All we'd need to do is solve for $L \mathbf{v}=\mathbf{i}$ (where $\mathbf{i}$ represents the vector of currents)

Now, let us try computing the value of the effective resistance between nodes 1 and $n$ (i.e. $R_{1 n}$. We can approach this computation in two different ways

Method 1: Solve

$$
\text { (*) } L\left[\begin{array}{c}
1 \\
V_{2} \\
\vdots \\
V_{n-1} \\
0
\end{array}\right]=\left[\begin{array}{c}
i \\
0 \\
\vdots \\
0 \\
-i
\end{array}\right] \quad i=\frac{V}{R}, \quad V=1, \quad R=\frac{1}{i}
$$

and return $\frac{1}{i}$
$(*)$ This is called a boundary valued problem. In our case, $V_{1}$ and $V_{n}$ are the boundary; $\left(V_{1}, \ldots, V_{n}\right)$ is called harmonic, since all of the interior $V_{a}$ can be written as a convex combination of its neighbors.

Maximum Principle: If $f$ is harmonic, then its min and max are on the boundary.

Proof. If $V$ is interior, then there must exists neighbors $V_{a}$ and $V_{b}$ such that $V_{a} \leq V \leq V_{b}$.

Uniqueness Principle: If $f$ and $g$ are harmonic with the same boundary values, then $f=g$
Proof. $f-g$ is harmonic, with zero's at both boundaries, hence $f-g \equiv 0$, therefore $f=g$.

Method 2: Solve

$$
L \mathbf{v}=\left[\begin{array}{c}
1 \\
0 \\
\vdots \\
0 \\
-1
\end{array}\right]
$$

and return $R_{1 n}=V_{1}-V_{n}$ (How do we know that an assignment of $\mathbf{v}$ exists?)

Another way to view the Laplacian: Boundary Operator (vertex-edge matrix): $B^{n \times m}$ and pick a direction to orient each edge. Consider the following example


Let $C_{1}, \ldots, C_{m}$ denote the conductances of $e_{1}, \ldots, e_{m}$

$$
\mathbf{C}=\left[\begin{array}{lll}
C_{1} & & 0 \\
& \ddots & \\
0 & & C_{m}
\end{array}\right]
$$

Note that if $f$ is a flow then $B f$ will equal, for each vertex, the surplus flow at that vertex. Thus $-B f$ will be the needed injected flow at each vertex.

Observe that

- $B^{T} \mathbf{v} \equiv$ the voltage drop across each edge
- $\mathbf{C} B^{T} \mathbf{v} \equiv$ minus the current flow along each edge
- $\mathbf{C}-B^{T} \mathbf{v} \equiv$ the current flow along each edge by Ohm's Law.
- $(-B) \mathbf{C}\left(-B^{T} \mathbf{v}\right) \equiv$ the need injected current at each vertex

This gives us that $L=B \mathbf{C} B^{T}$
If $G$ is a connected graph, we're interested in answering the following questions:

- What is $\operatorname{rank}(L)$ ?
- What is $\operatorname{ker}(L)$ ?

Consider

$$
x^{T} L x=x^{T} B \mathbf{C} B^{T} x=\left(B^{T} x\right)^{T} \mathbf{C} B^{T} x=\sum_{(a, b) \in E} \mathbf{C}_{a b}\left(x_{a}-x_{b}\right)^{2}
$$

thus

$$
x^{T} L=0 \Longleftrightarrow \forall(a, b) \in E \quad\left(x_{a}-x_{b}\right)^{2}=0=\left(x_{a}-x_{b}\right)
$$

Therefore if $G$ is connected $\Longrightarrow \forall a, b \quad x_{i}=x_{j}$
Hence, the kernel of $L$ is $\left.\left\langle\begin{array}{c}1 \\ \vdots \\ 1\end{array}\right]\right\rangle$ while the rank is $n-1$
Claim 1. Lx $=0$ iff $x^{T} L x=0$
Proof. The forward direction is clear, the backwards direction is

$$
x^{T} L x=x^{T} B C B^{T} x=\left(\mathbf{C}^{1 / 2} B^{T} x\right)^{T}\left(\mathbf{C}^{1 / 2} B^{T} x\right)=0
$$

which means that $\mathbf{C}^{1 / 2} B^{T} x=0$ which means that $B \mathbf{C}^{1 / 2} \mathbf{C}^{1 / 2} B^{T} x=0$, hence $L x=0$

### 1.3 Current and Energy/Power Dissipation



Newton said that

$$
\begin{aligned}
\text { Energy } & \equiv \text { Force } \cdot \text { Speed } \\
& \equiv \text { Volt } \cdot \text { Current } \\
& \equiv V i \\
& \equiv C V^{2} \equiv i^{2} R
\end{aligned}
$$

In our network, this means that Energy $=\frac{1}{2} \sum_{x, y}\left|i_{x y}\right| \cdot\left|\mathbf{v}_{x}-\mathbf{v}_{y}\right|$

$$
v^{T} L v=v^{T} B \mathbf{C} B^{T} v=\left(B^{T} v\right)^{T} \mathbf{C}\left(B^{T} v\right)=\sum_{\text {oriented }(x, y) \in E} \mathbf{C}_{x y}\left(V_{x}-V_{y}\right)^{2}=\text { Energy }
$$

### 1.4 Flows

We define two types of flows in our network:
Definition 2. A flow is a function $f: E \rightarrow \mathbb{R}$ over oriented edges
Definition 3. potential flow $=\left\{\boldsymbol{C} B^{T} \boldsymbol{v} \mid \boldsymbol{v} \in \mathbb{R}^{n}\right\} \equiv P_{G}$
Definition 4. circulation flows $=\left\{f \in \mathbb{R}^{m} \mid B f=0\right\} \equiv C_{G}$
Assume $G$ is connected and we are given its spanning tree $T$.
Claim 2. $C_{G}$ is a subspace, and the $\operatorname{dim}\left(C_{G}\right)=m-n+1$
Note that:

- Proving that this is a subspace is easy.
- $E \backslash T$ denotes the non-tree edges of $G$, hence $|E \backslash T|=m-n+1$
- Any flow on $E \backslash T$ can be extended to $C_{G}$ on $G$ (to be shown in homework)
- $f, g \in C_{G}$ and $f \backslash T=g \backslash T$ then $f=g$

Claim 3. The dimension of the subspace $P_{G}$ is $n-1$
Claim 4. $f_{C} \in C_{G}$ and $g_{P} \in P_{G}$ then $f_{C}^{T} R g_{P}=0$ where $R=\left[\begin{array}{ccc}R_{1} & & 0 \\ & \ddots & \\ 0 & & R_{m}\end{array}\right]$
Proof. $\exists v$ such that $g_{P}=C B^{T} \mathbf{v}$. Now we have

$$
f_{C}^{T} R g_{P}=f_{C}^{T} R C B^{T} \mathbf{v}=f_{C}^{T} B^{T} \mathbf{v}=\left(B f_{C}\right)^{T} \mathbf{v}=0^{T} \mathbf{v}=0
$$

(observe that we use the fact that $R C=I$ )
Therefore, $C_{G} ; P_{G}$ spans $\mathbb{R}^{m}$ (all flows) i.e. $\forall f \in \mathbb{R}^{m}, \exists!f_{C}, f_{P}$ such that $f=f_{C}+f_{P}$
Definition 5. $f_{a}=\sum_{a \neq b} f_{a b}$ where $a, b \in V$

Definition 6. $f$ is a unit flow from a to $b$ if:

- $f$ is a flow
- $f_{a}=1$ and $f_{b}=-1$
- $f_{x}=0$ for $x \neq a, b$

Thomson's Principle: If the following two conditions hold, then $f^{T} R f \leq g^{T} R g$

- $f$ is unit potential flow from $a$ to $b$
- $g$ is any unit flow from $a$ to $b$

Proof. We know that $g=f+f_{c}$ where $f_{C} \equiv$ circulation

$$
g^{T} R g=\left(f+f_{C}\right)^{T} R\left(f+F_{C}\right)=f^{T} R f+2 f_{C} R f+f_{C}^{T} R f_{C}=f^{T} R f+f_{C}^{T} R f_{C} \geq f^{T} R f
$$

Thus Thomson Principle gives yet another equivalent definition effective resistance is:
Definition 7. The effective resistance from a to $b\left(E R_{a b}\right)$ can be defined as $f_{P}^{T} R f_{p}$, where $f_{p}$ denotes the unit potential flow from a to $b$.

Rayleigh's Monotonicity Law: If $\bar{R} \geq R$ then $\overline{E R}_{a b} \geq E R_{a b}$
Proof. Let $f$ be the unit potential flow in $G_{R}$, and $g$ be the unit potential flow in $G_{\bar{R}}$

$$
\begin{aligned}
\overline{E R}_{a} b=g^{T} \bar{R} g & =\sum_{e \in G} g_{e}^{2} \bar{R}_{e} \\
& \geq \sum_{e \in G} g_{e}^{2} R_{e} \\
& \geq \sum_{e \in G} f_{e}^{2} R_{e} \quad \text { (by Thomson) } \\
& =f_{e}^{T} R f_{e}=E R_{a b}
\end{aligned}
$$

These two results allow us to show that $R_{a b}$ is a metric. That is:
HW: Show that $\left.R_{( } a b\right)$ is a metric space, i.e.,

- $R_{a b} \geq 0$
- $R_{a b}=0$ iff $a=b$
- $R_{a b}=R_{b a}$
- $R_{a c} \leq R_{a b}+R_{b c}$


## 2 Random Walks

Let $G=(V, E, w)$ be a (possibly directed) graph where

$$
\begin{aligned}
w_{a}=w(a) & \equiv \sum_{(a, b) \in E} w_{a b} \\
P_{a b} & \equiv \frac{w_{a b}}{w_{a}}
\end{aligned}
$$

Definition 8. Random walk on $G$ : Suppose at a given time we are at $a \in V$, we move to $b$ with probability $P_{a b}$

Ex. Let $V$ be all orderings of a deck of 52 cards. $P_{a b}$ will be the probability of going from some order $a$ to an order $b$ in one shuffle.

Fun question: Why do professionals play after 5 shuffles?
We can consider two views of a random walk:

- Particle view (the definition above)
- Wave view: there's a large number of simultaneous independent walkers

$$
\begin{gathered}
x^{(i)} \equiv \text { distance at time } i \\
x^{(i+1)}=A D^{-1} x^{(i)}
\end{gathered}
$$

Definition 9. Access Time, or Hitting Time $H_{a b}$ is the expected time to visit bstarting at a.
Definition 10. Commute Time: $K_{a b}=H_{a b}+H_{b a}$
Definition 11. Cover Time is the expected time to visit all nodes (we take the max over all starting nodes)

Definition 12. Mixing Rate: TODO in a future lecture

### 2.1 Random Walks on Symmetric Graphs

Idea: View a random walk as a walk on a network of conductors.
Input: $G=(V, E, C)$ where $C(a, b)=C(b, a)$
Consider a random walk starting at $x$ and ending at $b$. Let $h_{x}$ be the probability we visit $a$ before $b$ when starting at $x$, where $a \neq b$. Consider the following example:


We know that $h_{a}=1$ and $h_{b}=0$. What if we want to compute $h_{2}$ ? We know that this quantity must be greater than 0.5 . After some calculations, we can observe that $h_{x}=0.75$.

Example 2:


Claim 5. $h_{x}=\sum_{y} P_{x y} h_{y}$
We know that $P_{x y} \geq 0$ and that $\sum_{y} P_{x y}=1$. This means that $h_{x}$ is a convex combination of its neighbors.(Also, $h$ is harmonic with boundary $a, b!$ ).
Now, lets construct an identical electrical problem. Consider $V_{a}=1$ and $V_{b}=0 . \forall x \neq a, b$, $V_{x}=\sum_{y} \frac{C_{x y}}{C_{x}} V_{y}$. Observe that $\frac{C_{x y}}{C_{x}}=P_{x y}$, which means $h$ and $V$ equal.

Theorem 1. Set $V_{a}=1$ and $V_{b}=0$; Let $x \neq a, b$ "float" and then let $V_{x}$ be the probability that we visited a before $v$. The residual current at $x$ will be 0 .


In the above example, $h_{c}$ equals 0.75 , and $a=v_{1}$ while $b=v_{n}$. Algebraically we get

$$
L\left[\begin{array}{c}
1 \\
* \\
\vdots \\
* \\
0
\end{array}\right]=\left[\begin{array}{c}
* \\
0 \\
\vdots \\
0 \\
*
\end{array}\right]
$$

In general, we can have multiple sinks and goals:


We can compute this with one Laplacian solve.

### 2.2 Interpretation of Current as Random Walk

Consider one unit of potential current flow from $a$ to $b$, say $i$. What does $i_{x y}$ correspond to in a random walk from $a$ to $b$ ?

Theorem 2. $i_{x y}$ will be the expected net number of traversals of edge $(x, y)$ in a random walk from $a$ to $b$.

Proof. Let $U_{x}$ be the expected number of visits to $x$ before reaching $b$ starting at $a$. For HW, show that $\sum_{y} U_{y} P_{y x}$ (Note: $\sum_{y} P_{y x} \neq 1$ ). Now, recall that $C_{x}=\sum_{y} C_{x y}$, and note that

$$
C_{x} P_{x y}=C_{x}\left(\frac{C_{x y}}{C_{x}}\right)=C_{x y}=C_{y x}=C_{y}\left(\frac{C_{y x}}{C_{y}}\right)=C_{y} P_{y x}
$$

Thus,

$$
U_{x}=\sum_{y} U_{y} \frac{C_{y} P_{y x}}{C_{y}}=\sum_{y} U_{y} \frac{P_{x y} C_{x}}{C_{y}}
$$

Therefore, $\frac{U_{x}}{C_{x}}=\sum_{y} P_{x y}\left(\frac{U_{y}}{C_{y}}\right)$

Here, the voltage $V_{x}=\frac{U_{x}}{C_{x}}$, and its recurrence is $V_{x}=\sum_{y} P_{x y} V_{y}$ (and the residual current at $x$ is 0 ). This means that $V_{x}$ is harmonic with boundary conditions: $V_{b}=0$ and $V_{a}=\frac{U_{a}}{C_{a}}$ for the "correct" $U_{a}$
Define $j_{x y}$ as the current on edge $(x, y)$, and observe that

$$
j_{x y}=\left(V_{x}-V_{y}\right) C_{x y}=\left(\frac{U_{x}}{C_{x}}-\frac{U_{y}}{C_{y}}\right) C_{x y}=U_{x}\left(\frac{C_{x y}}{C_{y}}\right)=U_{x} P_{x y}-U_{y} P_{y x}
$$

Here, $U_{x} P_{x y}$ is the expected number of traversals from $x$ to $y$ (similarly, $U_{y} P_{y x}$ is the expected number of traversals from $y$ to $x$ ). This means that $j_{x y}$ is the expected net number of traversals from $x$ to $y$.

Now, we wish to show that the net current flow is 1 , i.e. $\sum_{y} j_{a y}=1$. This value must be 1 , since we must have had to leave $a$ once, for good, to get to $b$ (which means we never traveled that edge back).

Now, we are interested in computing $U_{a}$. Consider $a$ as the first vertex, while $b$ is the $n$th vertex. Solve

$$
L \mathbf{v}=\left[\begin{array}{c}
1 \\
0 \\
\vdots \\
-1
\end{array}\right]
$$

Set

$$
\mathbf{v}^{\prime}=\mathbf{v}-V_{n}\left[\begin{array}{c}
1 \\
0 \\
\vdots \\
1
\end{array}\right]
$$

i.e. set $V_{n}=0$. Hence $V_{1}=\frac{U_{1}}{C_{1}} \Longrightarrow U_{1}=V_{1} C_{1}$, which means we have found $U_{a}$ ! Also, $V=E R_{a b}$ and $U_{a}=C_{a} \cdot E R_{a b}$

### 2.3 How to compute hitting time

Recall that $H_{x b}$ denotes the expected time to reach $b$ from $x$. Let $H_{x}=H_{x b}$ for some fixed $b$. Consider the following recurrence:

$$
\begin{aligned}
& H_{b}=0 \\
& H_{x}=1+\sum_{y} H_{y} P_{x y} \quad(x \neq b)
\end{aligned}
$$

We can think of $H_{x}$ as a voltage $V_{x}$ :

$$
\begin{aligned}
V_{b} & =0 \\
V_{x} & =1+\sum_{y}\left(\frac{C_{x y}}{C_{x}}\right) V_{y}
\end{aligned}
$$

$$
\begin{aligned}
C_{x} V_{x} & =C_{x}+\sum_{y} C_{x y} V_{y} \\
C_{x} V_{x}-\sum_{y} C_{x y} V_{y} & =C_{x}
\end{aligned}
$$

Observe that the LHS is the graph Laplacian, while the RHS is the residual current. There are $n-1$ constraints, and by adding constraint for $v_{n}=b$

$$
\begin{gathered}
L \mathbf{v}-=\left[\begin{array}{c}
C_{1} \\
\vdots \\
C_{n-1} \\
\delta
\end{array}\right] \\
V_{n}=0 \quad C=\sum_{i} C_{i}
\end{gathered}
$$

where $\delta=C_{n}-C$
Now, if we wish to compute the hitting time from a $v_{x}$ to $v_{n}$, solve the above equation for $V_{x}$.
To solve for the commute time between $a\left(v_{1}\right)$ and $b\left(v_{n}\right)$, we have two methods:

1. Solve the following two

$$
L \mathbf{v}^{b}=\left[\begin{array}{c}
C_{1} \\
\vdots \\
C_{n}-C
\end{array}\right] \quad L \mathbf{v}^{a}=\left[\begin{array}{c}
C_{1}-C \\
\vdots \\
C_{n}
\end{array}\right]
$$

$H_{1 n}=V_{1}^{b}-V_{n}^{b}$ and $H_{n 1}=V_{n}^{a}-V_{1}^{a}$. Set $\mathbf{v}=\mathbf{v}^{b}-\mathbf{v}^{a}$ (and so $V_{i}=V_{i}^{b}-V_{i}^{a}$ ), and return

$$
K_{1 n}=H_{1 n}+H_{n 1}=V_{1}-V_{n}
$$

2. Solve the following

$$
L\left(\mathbf{v}^{b}-\mathbf{v}^{a}\right)=L \mathbf{v}^{b}-L \mathbf{v}^{a}
$$

This is equal to

$$
\left[\begin{array}{c}
C_{1} \\
\vdots \\
C_{n}-C
\end{array}\right]-\left[\begin{array}{c}
C_{1}-C \\
\vdots \\
C_{n}
\end{array}\right]=\left[\begin{array}{c}
C \\
\vdots \\
-C
\end{array}\right]=C\left[\begin{array}{c}
1 \\
\vdots \\
-1
\end{array}\right]
$$

Then, solve for

$$
L \mathbf{v}=\left[\begin{array}{c}
1 \\
\vdots \\
-1
\end{array}\right]
$$

And return $C\left(V_{1}-V_{n}\right)$ where $\left(V_{1}-V_{n}\right)=E R_{1 n}$
Method 2 motivates the following theorem:
Theorem 3. $K_{a b}=C \cdot E R_{a b}=2 m \cdot E R_{a b}$
For trees, this means that the commute time between two vertices is $2(n-1) \cdot E R_{a b}$


[^0]:    ${ }^{1}$ Originally $15-859 \mathrm{~N}$ notes by Tanvi Bajpai

