15-859NN
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Spectral Graph Theory

Yet another way to search a Graph

So far we have used

1) DFS
2) BFS
3) Today: Random Walk!

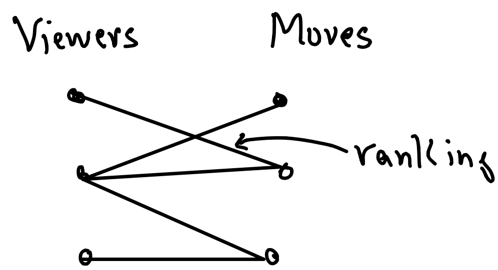
We will show Random Walk are intimately related to viewing the graph as a network of resistors.

This will allow us to use more math!

1) Linear algebra
    1) Linear systems
    2) Eigenvalues & vectors
    3) Numerical analysis
2) Matrix Chernoff bounds
3) Current & Energy

Resistive Model of a Graph & Random Walks

Modivation: Making a recommendation.
(Netflix)



Question: Should we recomend M to V?
$\equiv \text{Score}(V,M)$

Idea 1: Set $\text{Score}(V,M) = \dfrac{1}{\text{Graph dist}(V,M)}$

eg $w_{ij} = 1/\text{rank}_{ij}$

$$\text{Score}(V,M) = \frac{1}{\min\limits_{VPM} \text{Length}(P)}$$

Idea 2: Set $W(P) = \min\limits_{e \in P}(\text{rank}(e))$ $\qquad P \equiv \text{path}$

$$\text{Score}(V,M) = \max_{VPM} W(P)$$

<u>Problem</u> For ideas 1) & 2) extra paths may not improve score.

<u>Idea 3:</u> Score$(V, M) \equiv$ Max flow from V to M

<u>Prob</u> Shorter paths do not improve score.

<u>Idea 4</u> View edges as conductors

Score$(V, M) \equiv$ "effective conductance"

<u>Idea 5</u> Consider random walk from V to M.

$$\text{Score}(V, M) = \frac{1}{\text{hit}(V, M) + \text{hit}(M, V)}$$

where $\text{hit}(V, M) \equiv$ Expected length random walk from V to M.

We will show 4) & 5) are equal up to scaling

Hypothesis:

Effective Conductance &
Comute Time $\equiv Hit(v,u) + Hit(u,v)$
are better scores.

To Do:

1) Give formal definitions
2) Develop basic theory.
3) Give effecient algorithms
4) Find applications

A New Type of Flow

(Recall) The max flow problem: (Model)

1) Graph $G = (V, E)$
2) Edge Capacities C_i
3) Edge flow : $E \to \mathbb{R}$
4) Model or Constraints:
   1) flowin = flowout
   2) flow $\leq$ capacity (Defining Property)
5) Goal: Find Maximum flow

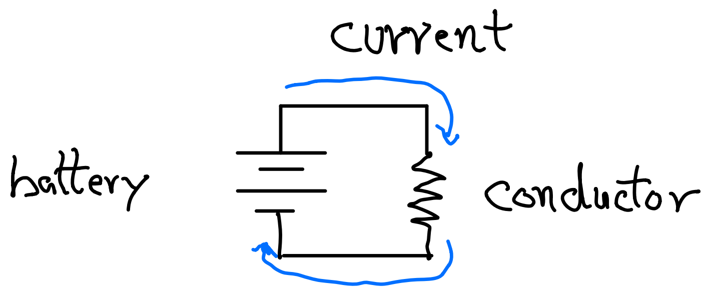
Electrical Flow Problem: (Model)

1) Graph $G=(V,E)$ (s,t)
2) Edge Conductances (Resistors)
3) a) Edge Currents (flows)
   b) Vertex Potentials (Voltages) (new)
4) Model or Constraints
   a) flowin = flowout ($v \notin \{s,t\}$)
   b) Flows & Potentials satisfing "Ohm's Law". new
5) Goal:
   1) Find vertex potential satisfing 4a & 4b with prescribed flow from s to t

   Equivalently

   2) Find min cost (energy) flow of fixed size from s to t.

Ohm's Law (Electrical View)



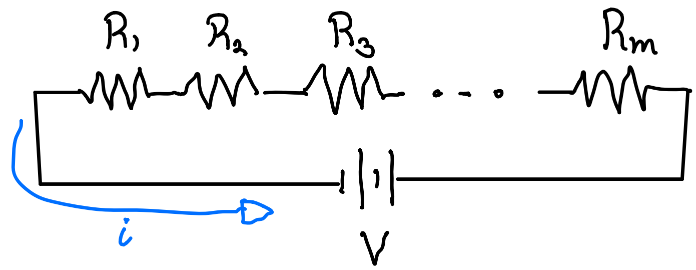
Electrical Model	Flow Model
$C \equiv$ conductance	capacity
$R = 1/C \equiv$ resistance	capacity
$i \equiv$ current	flow
$V \equiv$ voltage or potential	new

Ohm's Law: $i = C \cdot V$ or $iR = V$

(Voltage & current are linearly related)

Important Simple Identities

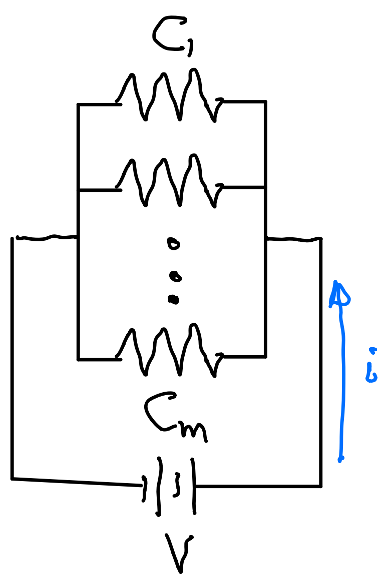
Resistors in Series Sum



(Graph Dist replacement)

$i = V/R$ where $R = R_1 + \cdots + R_m$

Conductors in Parallel Sum



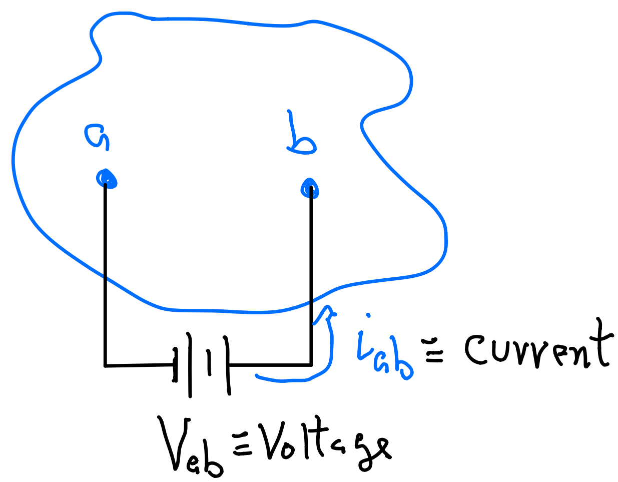
$i = V \cdot C$ where

$C = C_1 + \cdots C_m$

(Graph Max-Flow Replacement)

Effective Resistance/Conductance

Let $G=(V,E)$ be a network of resistors & $a,b \in V$



Claim: $\forall\, V_{ab}\ \exists!\, i_{ab}$

& i_{ab} is linear in V_{ab}

Def $R_{ab} = V_{ab}/i_{ab}$ (Effective Resistance)

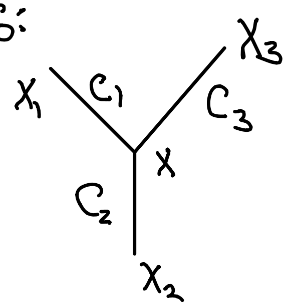
We need to show R_{ab} is well defined!

1) We will assume that current goes from high to low voltage.

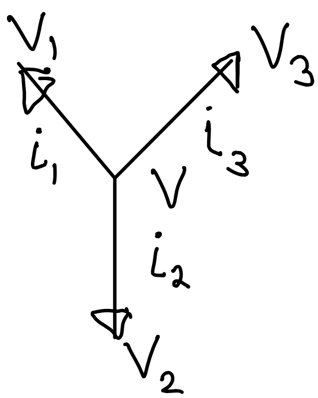
Computing Effective Resistance from a to b

Kirchhoff's Law: Except for a & b at each node flowin = flowout.

EG:



Voltages



Ohms Law

$i_1 = c_1(V - V_1)$

$i_2 = c_2(V - V_2)$

$i_3 = c_3(V - V_3$

residual current $= i_1 + i_2 + i_3$

Thus $i_1 + i_2 + i_3$ is needed injected current needed to satisfy Ohm.

By Kirchoff: $i_1 + i_2 + i_3 = 0$

$$C_1(V - V_1) + C_2(V - V_2) + C_3(V - V_3) = 0$$

$$(C_1 + C_2 + C_3)V = C_1V_1 + C_2V_2 + C_3V_3$$

Let $C = C_1 + C_2 + C_3$

then $CV = C_1V_1 + C_2V_2 + C_3V_3$

or $V = \frac{C_1}{C}V_1 + \frac{C_2}{C}V_2 + \frac{C_3}{C}V_3$

Thus V must be a convex combination of V_1, V_2, V_3

The needed current needed to satisfy Ohm's Law will be $CV - C_1V - C_2V_2 - C_3V_3$

Thus injecting current into node is positive. While removing current is negative.

We next determine the general case.

Graph Laplacian

We works out the general case

$G=(V,E,c)$ $\quad c: E \to \mathbb{R}^+$

$V=\{v_1,\cdots,v_n\}$

Let $d_i = d_i(v_i) = \sum_{(i,j)\in E} c_{ij}$

<u>Def</u> Laplacian$(G) = L(G) = L$

$$L_{ij} = \begin{cases} d(v_i) & \text{if } i=j \\ -c_{ij} & \text{if } (i,j)\in E \\ 0 & \text{o.w} \end{cases}$$

Weighted Adjacency Matrix A

$$A_{ij} = \begin{cases} c_{ij} & \text{if } (i,j)\in E \\ 0 & \text{o.w} \end{cases}$$

Degree Matrix $D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$

<u>Note</u> $L = D - A$

<u>Thus</u> $V \equiv$ vector of potentials then

$(LV)_i$ = needed injected current need to maintain V.

Method 1 for Computing Effective Resistance

Note If P is a column vector of potentials/voltage then $(LP)_i \equiv$ residual current at v_i.

Let $a = v_1$ & $b = v_n$

Solve $L \begin{pmatrix} 1 \\ v_2 \\ \vdots \\ v_{n-1} \\ 0 \end{pmatrix} = \begin{pmatrix} i \\ 0 \\ \vdots \\ 0 \\ -i \end{pmatrix}$ for $i, v_2, \ldots, v_{n-1}$ $(*)$

Return $1/i$ possibly ∞.

Claim $(*)$ is a boundary value problem and thus always has a unique solution.

(no proof)

$(*)$ is not in normal linear algebra form $Ax = b$!

Method 2 Solve $LV = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}$ return $R_{1n} = V_1 - V_n$

Does V exist? (When?)

Another way to view the Laplician.

Boundary Operator (vertex edge matrix)

$$B^{n\times m} \equiv \begin{array}{c} \\ V_1 \\ \\ \\ V_n \end{array} \begin{array}{ccc} e_1 & e_i & e_n \\ \hline & & \\ & -1 & \\ & 1 & \\ & & \\ \hline \end{array}$$

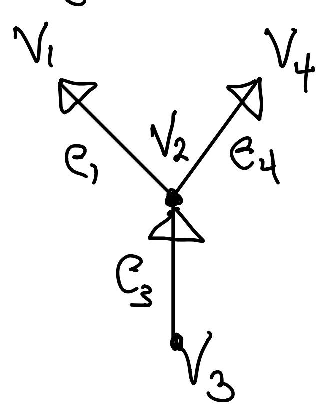
for each column e_i

1 at head

-1 at tail

0 o.w.

e.g.



$$B \equiv \begin{array}{c|ccc} & e_1 & e_3 & e_4 \\ \hline V_1 & 1 & 0 & 0 \\ V_2 & -1 & 1 & -1 \\ V_3 & 0 & -1 & 0 \\ V_4 & 0 & 0 & 1 \end{array}$$

Note: If f is a vector of flows then
Bf is a vector of residual flow at vertices.

Writing L as MM^T

Note: If f is a flow then
$Bf \equiv$ surplus flow at vetices
$-Bf \equiv$ injected flow (*)

Let $e_1, \dots, e_m$ be the edges of G
and $c_1, \dots, c_m$ their conductance.

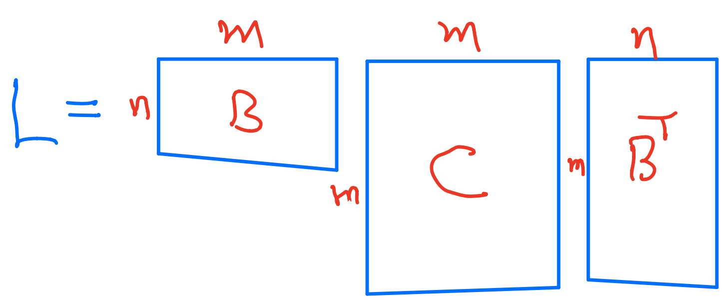
Def $C = \begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_m \end{pmatrix}$

Consider: BCB^T
Let V be a vector of voltages

Observe

1) $B^T V \equiv$ Voltage drop across each edge
2) $CB^T V \equiv$ minus the current flow on each edge. (Ohm's)
3) $-CB^T V \equiv$ Current flow
4) $(-B)(-CB^T)V \equiv$ inject current

Thus $BCB^T = (-B)(-CB^T) = L$



$(m > n)$

Assume G is a connected graph.

Questions:

what is $\text{rank}(L_G)$?

what is $\ker(L_G) = \{x \mid L_G x = 0\}$?

Consider $x^T L x = x^T B C B^T x$

$$= (B^T x)^T C B^T x = \sum_{(i,j)\in E} C_{ij} (x_i - x_j)^2$$

thus

$x^T L x = 0$ iff $\forall (i,j) \in E \; (x_i - x_j)^2 = 0 = (x_i - x_j)$

(why?)

G connected $\Rightarrow \forall i,j \; x_i = x_j$

$\ker(L_G) = \left\langle \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\rangle$

$\text{rank}(L) = n-1$

<u>Note:</u> $Lx = 0$ iff $x^T L x = 0$

$(\Rightarrow)$ clear

$(\Leftarrow)$ $x^T L x = x^T B C B^T x = (C^{1/2} B^T x)^T (C^{1/2} B^T x) = 0$

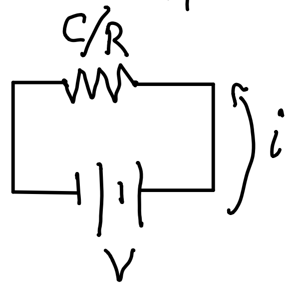
$$\Rightarrow C^{1/2} B^T x = 0$$

$$\Rightarrow B C^{1/2} C^{1/2} B^T x = 0$$

$$\Rightarrow Lx = 0$$

Current & Energy/Power Dissipation.

Model



Post Newton: Energy $\equiv$ Force $\cdot$ Distance $\equiv$ Force $\cdot$ Speed

Electrical Energy $\equiv$ Volt $\cdot$ Current

$$\equiv V \cdot i$$

$$i = CV \quad (1)$$

$$\equiv CV^2$$

$$V = iR \quad (2)$$

$$\equiv i^2 R \quad (3)$$

For Network $E = \frac{1}{2} \sum_{x,y} |i_{xy}| \, |V_x - V_y|$

Energy in terms of voltages

$$V^T L V = V^T B C B^T V = (B^T V)^T C (B^T V)$$

$$= \sum_{\text{oriented } (x,y)\in E} C_{xy} (V_x - V_y)^2 = E$$

$$= \sum_{(x,y)\in E} |i_{xy}|\,|V_x - V_y| \overset{(1)}{=} E$$

Energy of a flow

$$E(f) \underset{\text{by (3)}}{=} \sum R_i f_i^2 = f^T R f \qquad R = \begin{pmatrix} R_1 & & 0 \\ & \ddots & \\ 0 & & R_n \end{pmatrix}$$

Two Types of Flow

Recall: A flow $f: E \to \mathbb{R}$ (oriented edges)

Thus the set of all flow is a m-dim vector space

Def Potential flows $\equiv \{ CB^T v \mid v \in \mathbb{R}^n \} \equiv P_G$

If G is connected then P_G is a $(n-1)$-dim space.

Def Circulation flows $\equiv \{ f \in \mathbb{R}^m \mid Bf = 0 \} = C_G$

G is connected & Let T be a spanning tree.

Claim C_G is a $(m-n+1)$-dim space (G connected)

1) Subspace (easy)
2) $E \setminus T \equiv$ nontree edges
   $|E \setminus T| = m-n+1$
3) Claim any flow on $E \setminus T$ can be extended to a circulation.
4) $f, g \in C_G$ then $f \restriction E \setminus T = g \restriction E \setminus T$ iff $f = g$.

$\mathrm{Dim}(P_G) = n-1$

$\mathrm{Dim}(C_G) = m-n+1$

R orthogonality of potential & Circulation flows

<u>Thm</u>: Let $f_C \in C_G$ & $g_P \in P_G$ then

$$f_C^T R\, g_P = 0 \quad \text{where} \quad R = \begin{pmatrix} R_1 & & 0 \\ & \ddots & \\ 0 & & R_n \end{pmatrix}$$

That is, $\sum R_i f_i \cdot g_i = 0$

<u>pf</u> $g_P \in P_G \Rightarrow \exists V$ st $g_P = C B^T V$ thus

$$f_C^T R\, g_P = f_C^T R C B^T V = f_C^T B^T V$$
$$= (B f_C)^T V = 0^T V = 0$$

<u>Thus</u>: 1) C_G & P_G spans R^m (all flows)

2) $\forall f \in R^m\ \exists!\ f_C$ & g_P st $f = f_C + g_P$

<u>Def</u>: $f_a = \sum_{(a,b) \in E} f(a,b)$

<u>Def</u>: $f \in R^m$ is a <u>unit-flow</u> from a to b if:

1) f is a flow
2) $-f_a = f_b = 1$
3) $f_c = 0$ for $\forall c \neq a, b$

Thomson's Principle

(Potential flows are min energy)

Thm: 1) If f is a unit potential flow from a to b.
2) If g is any unit flow from a to b.
then $f^T R f \le g^T R g$

Pf We know that $g = g_p + g_c$ where

1) g_p is a unit potential flow from a to b.
2) g_c is a circulation

But there is only one unit potential flow from a to b.
Thus $g_p = f$ & $g = f + g_c$

$$g^T R g = (f+g_c)^T R (f+g_c) = f^T R f + \underbrace{\cancel{2 g_c R f}}_{0} + g_c^T R g_c$$

$$= f^T R f + g_c^T R g_c \ge f^T R f.$$

Def or Thm Effective resistance from a to b

$$ER_{ab} = f_p^T R f_p$$

$f_p \equiv$ unit potential flow from a to b.

Rayleigh's Monoticity Law

(Adding new resistors only decreases resistance)

Thm: Suppose networks $G_R = (V, E, R)$, $G_{\bar{R}} = (V, \bar{E}, \bar{R})$ s.t. $\bar{R} \geq R$ then $\overline{ER}_{ab} \geq ER_{ab}$

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pf Let $f \equiv$ unit potential flow from a to b in G_R

$g \equiv$ " " $G_{\bar{R}}$

$$\overline{ER}_{ab} = g^T \bar{R} g = \sum_{e \in G} g_e^2 \bar{R}_e$$

$$\geq \sum_{e \in G} g_e^2 R_e$$

$$\geq \sum_{e \in G} f_e^2 R_e \quad \text{(Thomson)}$$

$$= f_e^T R f_e = ER_{ab}$$

(HW) Show that R_{ab} is a metric space

i.e. 1) $R_{ab} \geq 0$

2) $R_{ab} = 0$ iff $a = b$

3) $R_{ab} = R_{ba}$

4) $R_{ac} \leq R_{ab} + R_{bc}$

<u>Question</u>: Are all finite metrics the resistive metric of some graph.