Spectral Graph Theory

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The Interplay between:

1) Graph Theory
2) Linear Algebra
3) Numerical Computing

Upside: Faster Algorithms

1) Single Source Shortest Paths
   (possibly negative edge weights)

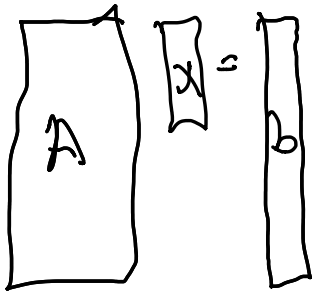
Bellman-Ford: $O(m \cdot n)$ time

m = #edges & n = #vertices

2016: New Method $O(m^{10/7})$

Regression Probs:

Simplest Case: Over Constrained System

e.g. 

Goal $\min_x \|Ax-b\|_2^2$

Standard Answer.

Solve $A^TA = A^Tb$

Prob: Computing A^TA is $O(mn^2)$

New Methods: Find B st $B^TBx' = B^Tb$

then $x \approx x'$

Faster than $O(mn^2)$ time.

We will introduce leverage scores!

Graph Sparsifiers

Input: Graph $G=(V,E)$

Output: Weighted subgraph $H \subseteq G$

s.t. $\forall S \subseteq V$

$$Cut_G(S,\bar{S}) \approx Cut_H(S,\bar{S})$$

Known bounds on number of edges in H.

1) $O(n \log n)$ edges Will do!

2) $O(n)$ edges Maybe do?

Topics We Hope to Cover

1) Graph as a network of Resistors.
2) Random walks and resistance
3) Matrix theory 101
    a) Eigenvalues & eigenvectors
        1) Spectral theorem
        2) Perron-Frobenius
        3) Courant-Fischer
4) Graph Cuts & eigenvalues
    a) Cheeger Thm
5) Estimating Eigenvalues
    a) Path embedding
    b) Hardy Inequality
6) Differential Equations & Matrix Exponentials.
7) Matrix Theory 102
    a) Singular values & vectors
        1) Weyl's Majorant Theorm.
    b) Matrix Chernoff Bounds
        1) Golden-Thompson

8) Graph Sparsifiers
   a) Sampling by effective resistance
9) Numerical Methods
   a) Basic Iterative Method
   b) Conjugate Gradiant
   c) Preconditioned Methods.
   d) Fast Laplacian solvers.
10) Understanding the fundamental Eigenvector.
   a) Fiedler's Thm.
11) Random Walks & Random Spanning Trees.
   a) Markov Chain Tree Theorm.
12) Approx Max Cut using Spectral Methods
13) Random Walks with resets.
14) Directed Laplacians
   1) Counting Trees
   2) Making them symmetric.
15) Low Stretch Spanning Trees.
16) Regression & Leverage Scores.