

Probability 101

15-859N
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Random Variable: Samples $\rightarrow \mathbb{R}$

For discrete Random Variable

the definition is straight forward.

Ex: Sample space = {H, T}; RV X

$$\text{Prob}[X=H] = 1/2$$

$$\text{Prob}[X=T] = 1/2$$

In the continuous case one
needs some care in general;

We will assume they are given by

i) Prob Density Fcn

ii) Cumulative Dist

The Exponential Distribution

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Prob Density Fcn

Def The Exponential RV X_β .

$$\text{Prob}[X_\beta = \mu] = \begin{cases} \beta e^{-\beta \mu} & \mu \geq 0 \\ 0 & \text{o.w} \end{cases}$$

More Formally:

$$\text{Prob}[X_\beta \in [u, u+dx]] = \begin{cases} \beta e^{-\beta u} dx & \mu \geq 0 \\ 0 & \text{o.w} \end{cases}$$

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Cumulative Dist

Def $F_\beta(y) \equiv \text{Prob}[X_\beta \leq y]$

For the Exponential

$$F_\beta(y) = \int_0^y \beta e^{-\beta x} dx \equiv \left[-e^{-\beta x} \right]_0^y = 1 - e^{-\beta y}$$

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Def Expected Value

$$E_x[X] = \int_{-\infty}^{\infty} y P_r[X=y] dy$$

Two ways to compute $E[X_\beta]$.

1) Def $E[X_\beta] = \int_0^\infty y \beta e^{-\beta y} dy = 1/\beta$

Using integration by parts.

2) $E[X_\beta] = \int_0^\infty \text{Prob}[X_\beta \geq y] dy$

For the Exponential

Now $\text{Prob}[X_\beta \geq y] = e^{-\beta y}$

thus $E[X_\beta] = \int_0^\infty e^{-\beta y} = -\frac{1}{\beta} e^{-\beta y} \Big|_0^\infty = 1/\beta$

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Order Statistics

Suppose X_1, \dots, X_n random variables

Def $X_{(k)} \equiv \text{Select}_k(X_1, \dots, X_n)$
 ie $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$

Suppose X_1, \dots, X_n are iid st

$$1) f(u) = \text{Prob}[X_i = u] \quad (\text{PDF})$$

$$2) F(u) = \text{Prob}[0 \leq X_i \leq u]$$

Then

$$\text{Prob}[X_{(1)} = u] = n (1 - F(u))^{n-1} f(u)$$

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If X_1, \dots, X_n are iid exponentials.

$$\text{Prob}[X_{(1)}=u] = n(e^{-\beta u})^{\frac{n-1}{n}} \beta e^{-\beta u} = n\beta e^{-n\beta u}$$

$$X_{(1)} \equiv \text{Exp}(n\beta) \quad \text{Thus } \text{Exp}[X_{(1)}] = \frac{1}{n\beta}$$

Conditional Probability

Suppose that A & B events

$$\text{Def } \text{Prob}[A|B] \equiv \text{Prob}[A \cap B] / \text{Prob}[B]$$

Memoryless Property

$$\text{Prob}[X_\beta > m+n | X_\beta > n] = \frac{e^{-\beta(n+m)}}{e^{-\beta n}} = e^{-\beta m}$$

Def Event A $\equiv X_\beta > m+n$ & Event B $\equiv X_\beta > n$

$$\text{Prob}[A \cap B] = \text{Prob}[A] = e^{-\beta(n+m)}$$

$$\text{Prob}[A \cap B] / \text{Prob}[B] = \frac{\text{Prob}[A]}{\text{Prob}[B]} = \frac{e^{-\beta(n+m)}}{e^{-\beta n}} = e^{-\beta m}$$

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Let $S_i = X_{(i+1)} - X_{(i)}$ for $i \geq 0$

By Memoryless property $S_i = \text{Exp}((n-i)\beta)$

Pf: Consider $S_1 = X_{(2)} - X_{(1)}$

Suppose $X_1 = X_{(1)}$ then

$$X_2, \dots, X_n \geq X_{(1)}$$

Thus $X_1 = X'_1 + X_{(1)}$ X'_1 new exponential

$$\text{Thus } \mathbb{E}[S_i] = \frac{1}{(n-i)\beta}$$

By linearity of expectation.

$$\begin{aligned} \mathbb{E}[X_{(n)}] &= \sum_{i=0}^{n-1} \mathbb{E}[S_i] = \frac{1}{\beta} \left(1 + \frac{1}{2} + \dots + \frac{1}{n} \right) \\ &= \frac{H_n}{\beta} \end{aligned}$$

$$\approx \frac{\ln n}{\beta}$$

$$\text{Thus } \mathbb{E}[X_{(1)}] = \frac{1}{n\beta}$$

$$\mathbb{E}[X_{(n)}] = \frac{\ln n + o(1)}{\beta}$$

Concentration for $X_{(n)}$?

$$\Pr \left[X_i \geq \frac{c \ln n}{\beta} \right] = e^{-\frac{\beta c \ln n}{\beta}} = e^{-c \ln n} = n^{-c}$$

by union bound we get.

$$\Pr \left[X_{(n)} \geq \frac{c \ln n}{\beta} \right] \leq n \cdot n^{-c} = \frac{1}{n^{c-1}}$$

thus

$$\Pr \left[X_{(n)} \geq \frac{2 \ln n}{\beta} \right] \leq \frac{1}{n}$$

Generating Dist or Random Variables

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Let X_f be the random variable of

PDF $f: \mathbb{R} \rightarrow \mathbb{R}^+$ where $\int_{-\infty}^{\infty} f(x)dx = 1$

Note Not clear that RV exist.

But we can ask if we have one can we generate more.

Def: $f, g \in \text{PDF's}$ with RV Variables X_f & X_g

we say $f \leq g$ if \exists dt process D s.t. $X_f = D(X_g)$

Let U be uniform RV with PDF u .

$$\text{ie } u(x) = \begin{cases} 1 & \text{if } x \in [0, 1] \\ 0 & \text{o.w.} \end{cases}$$

Let U_2 be uniform $[0, 2]$ then

$$X_{U_2} = 2X_U \text{ thus } U_2 \leq U$$

& $U \leq U_2$?

Generating Exponential Dist from Uniform

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$$\text{PDF} \equiv f(x) = \beta e^{-\beta x} \text{ for } 0 < \beta \text{ & } x \geq 0$$

$$F(x) = \int_0^x f(x) dx = 1 - e^{-\beta x}$$

Thus $F: [0, \infty] \rightarrow [0, 1]$ 1-1 & onto

We get that $F(X_f)$ is uniform $[0, 1]$

Thus: $U \leq f$ but we want $f \leq U$

Let's find F^{-1} ie

$$\text{Solve for } x \text{ in } Y = F(x) = 1 - e^{-\beta x}$$

$$\text{iff } e^{-\beta x} = 1 - Y$$

$$\text{iff } -\beta x = \ln(1 - Y)$$

$$\text{iff } x = -\frac{1}{\beta} \ln(1 - Y) \text{ but } 1 - Y \text{ is uniform } [0, 1]$$

$$X = \frac{-\ln(1 - U)}{\beta} = X_{\text{Exp}}$$

Thus $X_{\text{Exp}} \leq X_u$

Alg Given RV U uniform $[0, 1]$

$$\text{Return } \frac{-\ln U}{\beta}$$

Generating Normal Dist

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The PDF: $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2\sigma^2}$

Setting $\sigma=1$ we get Gauss's Unit Normal

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

What if we try computing CDF!

$$\text{if } F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-x^2/2} dx$$

Thm $F(x)$ is not an elementary fn!

Question: $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} dx ? = 1$

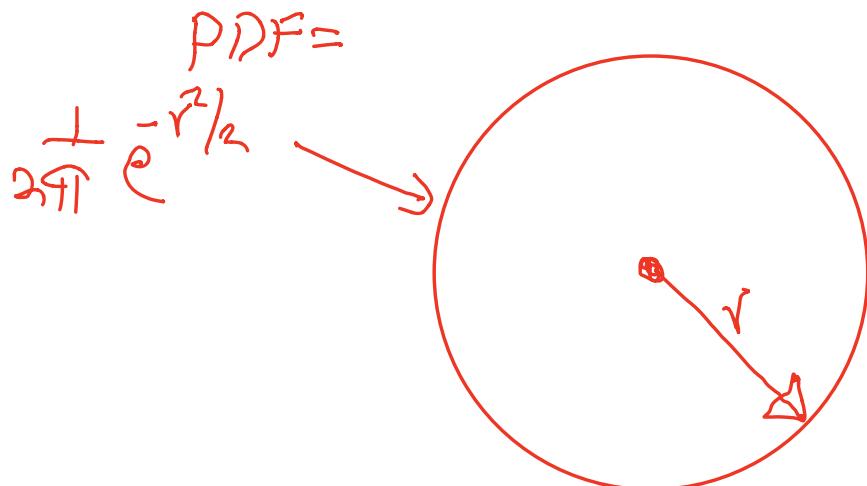
Note $f(x) = x e^{-x^2/2}$ is OK since

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$$\frac{d}{dx} \left(-e^{-x^2/2} \right) = x e^{-x^2/2}$$

Tricks compute 2D normal.

$$\begin{aligned} \text{Let } f(x,y) &= \frac{1}{2\pi} e^{-x^2/2} \cdot e^{-y^2/2} \\ &= \frac{1}{2\pi} e^{-(x^2+y^2)/2} \\ &= \frac{1}{2\pi} e^{-r^2/2} \text{ (in polar)} \end{aligned}$$



$$\text{In Polar: } f(r, \theta) = \frac{1}{2\pi} e^{-r^2/2} \quad (\text{Symmetric}) \quad 13$$

Let's compute the cumulative that is
Prob $f(r, \theta)$ is in a disk of radius r .

$$\begin{aligned} D(R) &= \int_0^R \frac{2\pi r}{2\pi} e^{-r^2/2} dr = \int_0^R r e^{-r^2/2} dr \\ &= -e^{-r^2/2} \Big|_0^R = 1 - e^{-R^2/2} \end{aligned}$$

$$\begin{aligned} y &= 1 - e^{-R^2/2} \\ e^{-R^2/2} &= 1 - y \\ -R^2/2 &= \ln(1-y) \\ R &= \sqrt{-2 \ln(1-y)} \\ R &= \sqrt{-2 \ln(y)} \end{aligned}$$

$$\begin{aligned} \text{Alg} \quad &\text{Pick } u, v \text{ uniform } [0, 1] \\ (\text{In polar}) \quad &\text{return } (\theta, r) \\ \text{where} \quad &\\ \theta &= 2\pi u \\ r &= \sqrt{-2 \ln(v)} \end{aligned}$$

$$\text{Or return: } \begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned}$$

Do we need to compute
 $\cos \theta$ & $\sin \theta$?

Yet another Normal Dist Alg 14

Let $B \equiv \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$

The unit ball

Note: $\text{Area}(B) = \pi$

Consider RV D

$$\text{Prob}[D = (x,y) \in B] = \frac{1}{\pi}$$

$$\text{Prob}[D = (x,y) \notin B] = 0$$

Alg UB[u,v] where U & V are uniform $[0,1]$

1) Set $U = 2U - 1$ & $V = 2V - 1$

"U & V are uniform $[-1,1]$ "

2) Set $W = U^2 + V^2$

3) Return (U,V) if $W \leq 1$

else restart (Try again)

Observe that

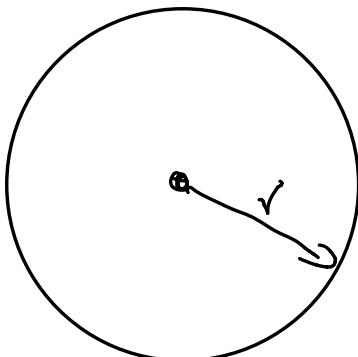
(U, V) is uniform over \mathcal{B} .

What can we say about w ?

Let's start with the RV \sqrt{w} .

Claim PDF of \sqrt{w} is $2r$.

Pf



$$\text{PDF}(w) = \frac{2\pi r}{\pi R^2}$$

$$\text{Thus } \text{CDF}(\sqrt{w}) = \int_0^R 2r dr = R^2$$

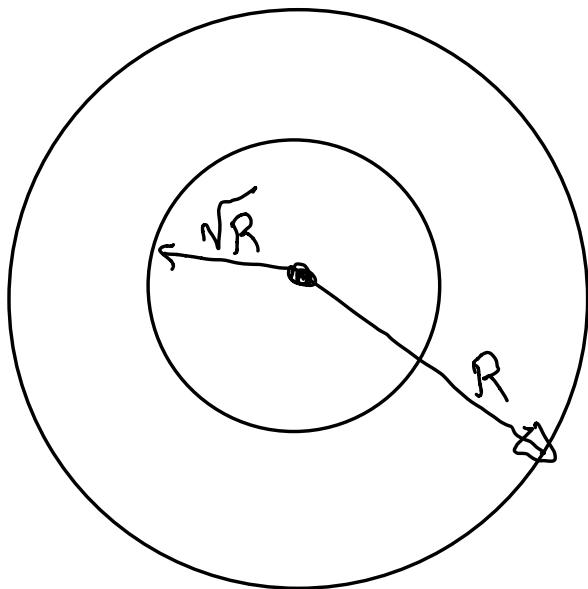
Consider: RV $w = (\sqrt{w})^2$

Question How do we get CDF of w

from \sqrt{w}

We look at the preimage

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$$CDF_w(R) = CDF_{\sqrt{w}}(\sqrt{R}) = R$$

Thus w is uniform $[0, 1]$

Note $(U, V)/\sqrt{w}$ is uniform on
unit circle

The Box-Muller Alg

Alg BM(u, v) where $u \& v$ are uniform $[0, 1]$.

1) Set $U = 2u - 1 \& V = 2v - 1$ (uniform $[-1, 1]$)

2) Set $W = U^2 + V^2$

if $W > 1$ then restart.

Note: (u, v) is uniform over unit disk.

3) Set $A = \sqrt{-2 \ln W}$ (scaling)

4) Return $T_1 = AU \& T_2 = AV$

Claim: BM generates 2D unit Gaussian.

After step 2

1) (U, V) is uniform over \mathcal{B} .

2) $(U/\sqrt{W}, V/\sqrt{W})$ is uniform
over unit circle.

3) W is uniform $[0, 1]$

Since w is $[0, 1]$ uniform

The Radial RV R is just

$$R = \sqrt{-2 \ln w}$$

For the angle RV Θ

we need $\cos \Theta, \sin \Theta$

But these are

$$U/\sqrt{w} \text{ & } V/\sqrt{w}$$

Thus we return

$$\left(\frac{R}{\sqrt{w}} U, \frac{R}{\sqrt{w}} V \right)$$

