

Graph Cuts & Eigenvalues

15-859N
3/3/21

2 Types of graph cuts.

Edge-cuts \equiv Remove edges to disconnect graph.

Vertex-cuts \equiv Remove vertices to disconnect graph.

This class only edge-cuts

Input: $G = (V, E, w)$ weighted undirected graph.

Return: A partition $S \subseteq V$ $\bar{S} = V \setminus S$

$$\text{Def } \text{Cap}(A, B) \equiv \sum_{\substack{(a, b) \in E \\ a \in A, b \in B}} w(a, b)$$

$$\text{Vol}(A) = \text{Cap}(A) \equiv \text{Cap}(A, V) = \sum_{a \in A} d_a$$

Cut Measures

2

1) Quotient cut or Isoperimetric number

$$g(G) = \min_{S \subseteq V} \frac{\text{Cap}(S, \bar{S})}{\min\{|S|, |\bar{S}|\}}$$

2) Sparest Cut

$$\alpha(G) = \min_{S \subseteq V} \frac{\text{Cap}(S, \bar{S})}{|S| \cdot |\bar{S}|}$$

Note $\frac{n}{2} \alpha(G) \leq g(G) \leq n \alpha(G)$

3) Conductance (Normalized Quotient Cuts)

$$\Phi(G) = \min_{S \subseteq V} \frac{\text{Cap}(S, \bar{S})}{\min\{\text{Cap}(S), \text{Cap}(\bar{S})\}}$$

4) Sparest Cut

$$\chi(G) = \min_{S \subseteq V} \frac{\text{Cap}(S, \bar{S})}{\text{Cap}(S) \cdot \text{Cap}(\bar{S})}$$

View Sparest Cut as a min cut of demands.

$$\text{Traffic or demand } (V_i, V_j) = d_i \cdot d_j$$

Claim $\text{Traffic}(S, \bar{S}) = \sum_{\substack{i \in S \\ j \notin S}} d_i \cdot d_j$

3

$$\text{bwt Traffic}(S, \bar{S}) = \left(\sum_{i \in S} d_i \right) \left(\sum_{j \in \bar{S}} d_j \right)$$

$$= \text{Cap}(S) \text{Cap}(\bar{S})$$

Def Demand(S, \bar{S}) \leq Traffic(S, \bar{S}).

more generally: Let $f: V \rightarrow \mathbb{R}^+$

Def Demand(v_i, v_j) $= f(v_i) \cdot f(v_j)$
(The product demand.)

$$\bar{\alpha}(G) \equiv \min_{S \subseteq V} \frac{\text{Cap}(S, \bar{S})}{\text{Demand}(S, \bar{S})}$$

Note: $\bar{\alpha}(G)$ is case $P_i = d_i / \sum d_i$

We can also view P_i as a mass.

$$\text{thus } \bar{\alpha}(G) = \min_{S \subseteq V} \frac{\text{Cap}(S, \bar{S})}{\min \{ \text{Mass}(S), \text{Mass}(\bar{S}) \}}$$

Main Cheeger Thms

4

Thm 1 $\lambda_2/2 \leq g(G) \leq \sqrt{2D}\lambda_2$ where

$$\lambda_2 = \lambda_2(L_G) \quad \& \quad D = \max_i d_i$$

Thm 2 $\lambda_2/2 \leq \bar{\chi}(G) \leq \sqrt{2\lambda_2}$ where

$\lambda_2 = \lambda_2(\bar{L})$ & $\bar{L} \equiv$ Normalized Laplacian of G
ie $\bar{L} = D^{-1/2} L D^{1/2}$

The upper bds on $g(G)$ & $\bar{\chi}(G)$ are called
the Cheeger direction

Suppose one has a Rayleigh $\frac{x^T L x}{x^T M x} = \lambda$ &
 $x^T M \bar{1} = 0$ then

Thm 3 $\bar{\chi}(G) \leq \sqrt{2 \max_i \left\{ \frac{d_i}{m_i} \right\} \lambda}$

Proof of easy direction of Thm 1

5

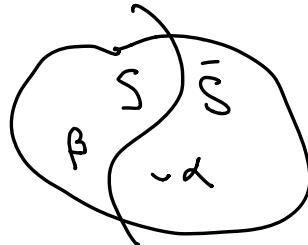
Claim $\lambda_2(L) \leq 2 \cdot g(G)$

that is \exists vector X s.t. $X^T 1 = 0$ $\frac{X^T L X}{X^T X} \leq 2 \cdot g$

$\stackrel{Pf}{\rightarrow}$ Let $S \subseteq V$ st $|S| \leq n/2$

Set $\alpha = |S|$ & $\beta = |\bar{S}|$

Let $X_i = \begin{cases} \beta & \text{if } V_i \in S \\ -\alpha & \text{o.w.} \end{cases}$



Note $X^T 1 = \beta \cdot |S| - \alpha \cdot |\bar{S}| = \alpha \beta - \alpha \beta = 0$

Let $g = \frac{\text{Cap}(S, \bar{S})}{|\bar{S}|}$

$$\text{Now: } \frac{X^T L X}{X^T X} = \frac{(\alpha + \beta)^2 \text{Cap}(S, \bar{S})}{\alpha \beta^2 + \beta \alpha^2} = \frac{n^2 \text{Cap}(S, \bar{S})}{n \alpha \cdot \beta}$$

$$= \left(\frac{\text{Cap}(S, \bar{S})}{\alpha} \right) \left(\frac{n}{\beta} \right) \leq 2 \cdot g$$

□

Proof of Upper Bd for Thm 2

6

We prove a stronger version:

Consider a spring-mass Laplacian $LX = \lambda MX$

$$\text{Let } S = \max_i \left(\frac{d_i}{m_i} \right)$$

Thm If $X^T \begin{pmatrix} m_0 \\ \vdots \\ m_n \end{pmatrix} = 0$ then $\underline{\Phi}(G) \leq \sqrt{2S \frac{X^T L X}{X^T M X}}$

$$\text{Here } \underline{\Phi}(G) = \frac{\text{Cap}(S, \bar{S})}{\min \{ \text{Mass}(S), \text{Mass}(\bar{S}) \}}$$

The proof is effective!

That is: Given X we find a "good" cut,
a "threshold cut".

Suppose $f: V \rightarrow \mathbb{R}$

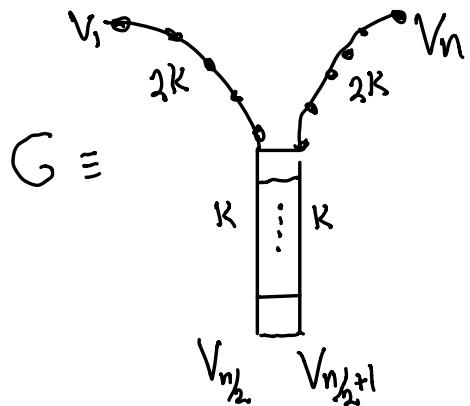
Sort V so that $f(v_i) \geq f(v_{i+1})$

Def: The i th threshold cut using $f \equiv S_i = \{v_1, \dots, v_i\}$

Note Threshold $S_{n/2}$ may be "bad"

7

Consider the Bug-Graph



Note G has an automorphism the reflection, τ .

Claim $Lf = \lambda_2 f$ then f is odd wrt τ

i.e. $f(v) = -f(\tau(v))$ (Aw)

Thus $\text{Cap}(S_{n/2}, \bar{S}_{n/2}) = K$ giving $\frac{1K}{3K} = \frac{1}{3}$

A better cut would be to remove one ear.

$$g \approx \frac{1}{2K} \approx \frac{3}{n}$$

The proof: Preliminaries $A \equiv \text{adj matrix}$

$$\text{Thus } L = D - A \text{ ie } z^T L z = \sum w_{ii} (z_i - z_j)^2$$

The Sum-Laplacian $P = D + A$

$$\underline{\text{Claim}} \quad z^T P z = \sum w_{ij} (z_i + z_j)^2$$

$$\underline{\text{Note}} \quad P = 2D - L$$

$$\underline{\text{Cauchy-Schwarz}} \quad (a^T b)^2 \leq (a^T a)(b^T b) \quad a, b \in \mathbb{R}^n$$

$$\begin{aligned} (z^T L z)(z^T P z) &= \left(\sum w_{ij} (z_i - z_j)^2 \right) \left(\sum w_{ij} (z_i + z_j)^2 \right) \\ &= \left(\sum w_{ij} |z_i - z_j|^2 \right) \left(\sum w_{ij} |z_i + z_j|^2 \right) \\ &\geq \left(\sum w_{ij} |z_i^2 - z_j^2| \right)^2 \end{aligned}$$

$$\text{Here } a = (\dots, \sqrt{w_{ij}} |z_i - z_j|, \dots)$$

$$b = (\dots, \sqrt{w_{ij}} |z_i + z_j|, \dots)$$

The Proof Let $z \in \mathbb{R}^n$ st

9

$$z^T M \bar{1} = 0 \quad \& \quad z_1 \geq z_2 \geq \dots \geq z_n$$

Assume $V = \{1, \dots, n\}$ & $S_i = \{1, \dots, i\}$

$$\text{Def} \quad \underline{\Phi}_i = \frac{\text{Cut}(S_i, \bar{S}_i)}{\min\{\text{Mass}(S_i), \text{Mass}(\bar{S}_i)\}}$$

$$\text{to show} \quad \exists i \quad \underline{\Phi}_i \leq \sqrt{2 \Delta \frac{z^T L z}{z^T M z}}$$

We modify z & G only decreasingly

$$\frac{z^T L z}{z^T M z} \quad \& \text{not changing } \underline{\Phi}_i$$

$$\text{Def} \quad \beta = \min \{ i \mid \text{Mas}(\bar{S}_i) \leq \text{Mas}(S_i) \}$$

two modifications to z & G .

1) shifting z st $z_\beta = 0$

2) $\forall (i, j) \in E$ st $z_i \cdot z_j < 0$ pinch (i, j) at β .

1) Shifting \underline{z} : $\underline{y} = \underline{z} + \alpha \underline{I}$

10

$$\text{Claim: } \frac{\underline{y}^T \underline{L} \underline{y}}{\underline{y}^T M \underline{y}} \leq \frac{\underline{z}^T \underline{L} \underline{z}}{\underline{z}^T M \underline{z}} \quad (\text{a})$$

$$\begin{aligned} \frac{\underline{y}^T \underline{L} \underline{y}}{\underline{y}^T M \underline{y}} &= \frac{(\underline{z} + \alpha \underline{I})^T \underline{L} (\underline{z} + \alpha \underline{I})}{(\underline{z} + \alpha \underline{I})^T M (\underline{z} + \alpha \underline{I})} \\ &= \frac{\underline{z}^T \underline{L} \underline{z}}{\underline{z}^T M \underline{z} + 2\alpha \underline{z}^T M \underline{I} + \alpha^2 \underline{I}^T M \underline{I}} \leq \frac{\underline{z}^T \underline{L} \underline{z}}{\underline{z}^T M \underline{z}} \end{aligned}$$

since $\underline{z}^T M \underline{I} = 0$ & $M \geq 0$

Pinching an edge at zero

For each edge st $y_i < 0 < y_j$



This give Laplaceian L'

11

Claim $y^T L' y \leq y^T L y$ (b)

each term $(y_i - y_j)^2$ where $y_i, y_j < 0$

is replaced with $(y_i - 0)^2 + (D - y_i)^2$

$$(y_i - y_j)^2 = y_i^2 - 2y_i y_j + y_j^2 \geq y_i^2 + y_j^2$$

Note The Masses are unchanged.

the degree at β may increase

$$\text{ie } D' \geq D$$

For sum-Laplaceians

$$\frac{y^T P' y}{y^T M y} = \frac{y^T (2D' - L') y}{y^T M y} \leq \frac{2y^T D' y}{y^T M y} = 2 \underbrace{\frac{y^T D y}{y^T M y}}_{y_\beta = 0}$$

$$\leq 2 \max_i \left(\frac{d_i}{m_i} \right) = 2 \mathcal{S}. \quad (\text{c})$$

$$2S \frac{\beta^T L \beta}{\beta^T M \beta} \cdot (\gamma^T M \gamma)^2 \stackrel{(C)}{\geq} (\gamma^T P' \gamma) \left(\frac{\beta^T L \beta}{\beta^T M \beta} \right) \gamma^T M \gamma$$

(a), (b) (CS)

$$\geq (\gamma^T P' \gamma) (\gamma^T L' \gamma) \geq \left(\sum w_{ij} |\gamma_i^2 - \gamma_j^2| \right)^2$$

thus

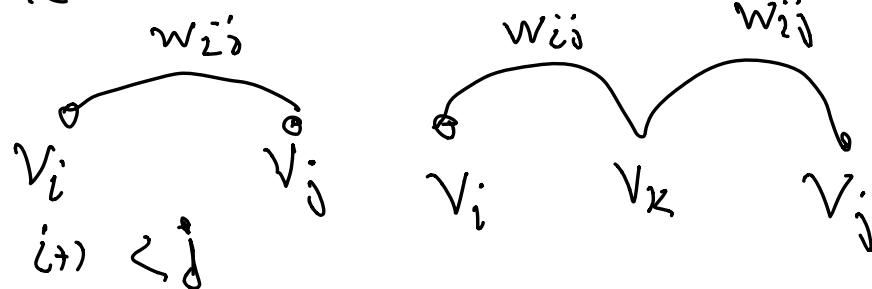
(sqrt)

$$\sqrt{2S \frac{\beta^T L \beta}{\beta^T M \beta}} (\gamma^T M \gamma) \geq \sum_{i < j \leq \beta} w_{ij} (\gamma_i^2 - \gamma_j^2) \quad (d)$$

$$+ \sum_{j > i \geq \beta} w_{ij} (\gamma_i^2 - \gamma_j^2)$$

Claim WLOG G is a weighted path graph (d is telescoping)

ie



Next Main Inequality of Proof

Recall $y_0^2 \geq y_1^2 \geq \dots \geq y_\beta^2$

Def $w_i = w_{i+1}$

$$\underline{\Phi}_i = \frac{w_i}{\text{Mas}(S_i)} \quad ; \quad \underline{\Phi}_{\text{left}}^* = \min_{i \leq \beta} \underline{\Phi}_i \quad ; \quad \text{Mas}(V_0) = 0$$

Consider

$$\begin{aligned} \sum_{i=1}^{\beta-1} w_i (y_i^2 - y_{i+1}^2) &= \sum_{i=1}^{\beta-1} \underline{\Phi}_i \text{Mas}(S_i) (y_i^2 - y_{i+1}^2) \\ &\geq \underline{\Phi}_{\text{left}}^* \sum_{i=1}^{\beta-1} \text{Mas}(S_i) (y_i^2 - y_{i+1}^2) \\ &= \underline{\Phi}_{\text{left}}^* \sum_{i=1}^{\beta-1} (\text{Mas}(S_i) - \text{Mas}(S_{i+1})) y_i^2 \\ &= \underline{\Phi}_{\text{left}}^* \sum_{i=1}^{\beta-1} m_i y_i^2 \\ &\leq \underline{\Phi}_{\text{left}}^* \gamma_{\text{left}}^T M \gamma_{\text{left}} \end{aligned}$$

14

$$\sqrt{2S \frac{\|L\|^2}{\|M\|^2}} (Y^T M Y)$$

$$\geq \underline{\Phi}_{\text{left}}^* Y_{\text{left}}^T M Y_{\text{left}} + \underline{\Phi}_{\text{right}}^* Y_{\text{right}}^T M Y_{\text{right}}$$

$$\text{Let } \underline{\Phi}^* = \min \{ \underline{\Phi}_{\text{left}}^*, \underline{\Phi}_{\text{right}}^* \}$$

$$\geq \underline{\Phi}^* (Y_l^T M Y_l + Y_r^T M Y_r)$$

$$= \underline{\Phi}^* (Y^T M Y)$$

thus

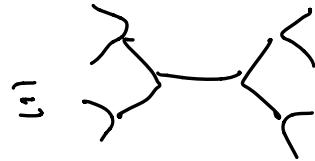
$$\sqrt{2S \frac{\|L\|^2}{\|M\|^2}} \geq \underline{\Phi}^* \quad \square$$

Two Extreme Graphs

15

$P_n \equiv$ Path Graph

$T_n \equiv$ Complete binary Tree



Thm $\lambda_2/2 \leq \overline{\lambda}(G) \leq \sqrt{2\lambda_2}$ where

We showed that $\lambda_2(P_n) = \Theta(1/n^2)$

HW shows that $\lambda_2(T_n) = \Theta(1/n)$

$$\overline{\lambda}(P_n) = \overline{\lambda}(T_n) = \frac{2}{n}$$

(*) is best possible for $\lambda_2(P_n)$ is $\sqrt{\lambda_2(P_n)} = \Theta(1/n)$

(*) is best possible for $\lambda_2(T_n)$ is $\lambda_2(T_n) = \Theta(1/n)$