

# Bounding Eigenvalues

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## Mediant of Fractions.

$$\text{Fractions} \equiv \left( \frac{a_1}{b_1}, \dots, \frac{a_n}{b_n} \right)$$

$$\text{Mediant} \equiv \frac{\sum a_i}{\sum b_i}$$

$$\text{Claim} \quad \frac{\sum a_i}{\sum b_i} \leq \max_i \frac{a_i}{b_i} \text{ for } \forall_i a_i, b_i > 0$$

picture pf

Consider points in  $\mathbb{R}^2$   $P_1 = (b_1, a_1) \dots P_n = (b_n, a_n)$

Let  $P \in \mathbb{R}^2$  be average or center of mass

$$P = \frac{1}{n} (\sum b_i, \sum a_i)$$

$$\text{Slope}(P_i) = \frac{a_i}{b_i} \quad \& \quad \text{Slope}(P) = \frac{\sum a_i}{\sum b_i}$$

by picture  $\text{Slope}(P) \leq \max_i \text{Slope}(P_i)$

## Courant-Fischer (CF)

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Thm (Courant-Fischer)  $A^{n \times n}$  sym

with eigens  $\mu_1 \geq \dots \geq \mu_n$  then

$$\mu_k = \max_{S \subseteq \mathbb{R}^n} \min_{x \in S} \frac{x^T A x}{x^T x}$$

$$\dim(S) = k$$

$$= \min_{T \subseteq \mathbb{R}^n} \max_{x \in T} \frac{x^T A x}{x^T x}$$

$$\dim(T) = n - k + 1$$

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View CF as a two person game.

Let  $x_1, \dots, x_n$  be the eigenvectors <sup>3</sup>  
for  $\mu_1 \geq \dots \geq \mu_n$

pf(CF) Case max-min for S

Case: ( $\mu_k \leq \max$ -min)

pick  $S = \langle x_1, \dots, x_k \rangle$  &  $x \in S$

$x = \sum c_i x_i$  for some  $c_1, \dots, c_k \in \mathbb{R}$

$$\frac{x^T A x}{x^T x} = \frac{\sum \mu_i c_i^2}{\sum c_i^2} \geq \frac{\mu_k c_k^2}{c_k^2} = \mu_k$$

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Case: ( $\mu_k \geq \max$ -min)

To show:  $\forall S \dim(S) = k$

$$\text{then } \min_{x \in S} \frac{x^T A x}{x^T x} \leq \mu_k$$

pf Let  $T = \langle x_k, \dots, x_n \rangle$

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Thus  $\dim(T) = n - k + 1$

$\therefore T \cap S \neq \emptyset$

$$\min_{x \in S} \frac{x^T A x}{x^T x} \leq \min_{x \in S \cap T} \frac{x^T A x}{x^T x}$$

$$\therefore x \in T \quad x = \sum_{i=k}^n c_i x_i$$

$$\frac{x^T A x}{x^T x} = \frac{\sum_{k \leq i \leq n} M_i c_i^2}{\sum c_i^2} \leq \frac{M_k c_k^2}{c_k^2} = M_k \quad \square$$

Let  $A$  adj matrix &  $D$  degree matrix

Let  $\mu_1 \geq \dots \geq \mu_n \in \lambda(A)$

$d_{\max}, d_{\text{ave}}$  be max & average degree

Lemma  $d_{\text{ave}} \stackrel{(a)}{\leq} \mu_1 \stackrel{(b)}{\leq} d_{\max}$

pf (a)

$$\mu_1 = \max_{x \neq 0} \frac{x^T A x}{x^T x} \geq \frac{\mathbf{1}^T A \mathbf{1}}{\mathbf{1}^T \mathbf{1}} = \frac{2m}{n} = d_{\text{ave}}$$

(b)

Let  $Ax = \mu_1 x$  &  $x_v \geq x_u \forall u \in V$

$$\mu_1 = \frac{(Ax)_v}{x_v} = \frac{\sum x_u A_{vu}}{x_v} \leq \sum A_{vu} = d_{\max}$$

Let  $x_1, \dots, x_n$  be the eigenvectors  $\} 3$

pf(CF) Case max-min for  $S$

( $\geq$ )

pick  $S = \langle x_1, \dots, x_k \rangle$  &  $x \in S$

$$x = \sum c_i x_i$$

$$\therefore \frac{x^T A x}{x^T x} = \frac{\sum u_i c_i^2}{\sum c_i^2} \geq \frac{u_k c_k^2}{c_k^2} = \mu_k$$

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( $\leq$ ) to show  $\forall S \dim(S) = k$

$$\min_{x \in S} \frac{x^T A x}{x^T x} \leq \mu_k$$

$A = \text{adj}(G)$  unit weights

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Lemma If  $G$  is connected then

$G$  is  $d$ -reg iff  $\mu_1 = d_{\max}$

$$\Leftrightarrow A \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = d \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$\Leftrightarrow Ax = d_{\max} x \quad \& \quad \max(x) = x_v$$

$$d x_v = \sum_{(v,u) \in E} x_u \Rightarrow x_u = x_v \quad \forall (v,u) \in E$$

connected  $\Rightarrow x_u = x_v \quad \forall u, v$

Thus WLOG  $x = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

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Lemma  $G$  is graph  $L = L(G)$

with eigenvalues  $0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$

then  $\lambda_n \leq 2d_{\max}$

pf  $L = D - A$

$$\lambda_n \leq \lambda_{\max}(D) - \lambda_{\min}(A)$$

$$\leq d_{\max} - (-d_{\max}) = 2d_{\max}$$

$A, B^{n \times n}$  Sym

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New Notation:  $A \succeq 0$  if  $\forall x \ x^T A x \geq 0$

In general:  $A \preceq B$  if  $\forall x \ x^T A x \leq x^T B x$   
ie.  $B - A \succeq 0$

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Note:  $A \preceq B$  &  $B \preceq C$  then  $A \preceq C$

$$A \preceq B \Rightarrow A + C \preceq B + C$$

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Graph  $G$  &  $H$   $G \preceq H$  if  $L_G \preceq L_H$

Thus  $G \preceq G \cup H$

$$\underline{\text{Thm}} \quad A \preceq B \Rightarrow \lambda_k(A) \leq \lambda_k(B)$$

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$$\underline{\text{pf}} \quad \lambda_k(B) = \min_{\dim(\mathcal{T})=n-k+1} \max_{x \in \mathcal{T}} \frac{x^T B x}{x^T x}$$

$$\geq \min_{\dim(\mathcal{T})=n-k+1} \max_{x \in \mathcal{T}} \frac{x^T A x}{x^T x}$$

$$= \lambda_k(A)$$

## Pseudo-Inverse

$$a \in \mathbb{R}$$

$$\text{Def } a^+ = \begin{cases} 0 & \text{if } a = 0 \\ a^{-1} & \text{o.w.} \end{cases}$$

$D$  is diagonal.

$$\text{Def } (D^+)_{ij} = D_{ij}^+$$

$$A \text{ is sym} \Rightarrow U^T \Lambda U$$

$$\text{Def } A^+ = U^T \Lambda^+ U$$

$$G = (V, E, w) \quad V = \{v_1, \dots, v_n\}$$

$$G_m = (V, E = \{(1, n)\}) \quad \begin{array}{c} v_1 \quad v_n \\ \text{---} \end{array}$$

$ER_{ij} \equiv$  effective resistance in  $G$ .

Consider arbitrary graph  $G$ ;  $|G|_n$

$$G_m \cong \begin{array}{c} \bullet \quad \bullet \\ v_1 \quad v_n \\ \text{---} \end{array}$$

Claim:  $G_m \preceq \alpha G$  iff  $ER_m \leq \alpha$

Claim:  $G_m \preceq \alpha G \Rightarrow ER_m \leq \alpha$

pf Let  $f$  be unit potential flow with potential  $V$ .

$$ER_m = f^T R f = V^T L V \geq \frac{V^T G_m V}{\alpha} = \frac{(ER_m)^2}{\alpha}$$

$$\Rightarrow \alpha \geq ER_m$$

$$(HW) \quad ER_m \leq \alpha \Rightarrow G_m \preceq \alpha G$$

## Upper Bounding $\lambda_2$

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$$(CF) \quad \lambda_2 = \min_{V^T \mathbf{1} = 0} \frac{V^T L V}{V^T V}$$

$$\text{thus } \forall X \text{ s.t. } X \perp \mathbf{1}, \quad \lambda_2 \leq \frac{X^T L X}{X^T X}$$

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Let  $P_n$  be the unit weight path graph.

$$X = (-\frac{n}{2}, -\frac{n}{2}+1, \dots, \frac{n}{2}-1, \frac{n}{2})$$

$$\text{Note } \mathbf{1}^T X = 0$$

$$X^T L_{P_n} X = \sum (X_i - X_{i+1})^2 = n-1$$

$$X^T X = 2 \sum_{i=1}^{n/2} i^2 \approx \frac{2 (n/2)^3}{3} = \frac{n^3}{12}$$

$$\text{Thus } \lambda_2 \leq \frac{n-1}{n^3/12} \approx \frac{12}{n^2}$$

What about a matching lower bd!

## Lower bounding $\lambda_2$ !

We start with the Complete graph  $K_n$ .

$$L(K_n) = \begin{pmatrix} n-1 & -1 & \dots & -1 \\ -1 & n-1 & & -1 \\ & & \ddots & \\ -1 & & & n-1 \end{pmatrix}$$

note

$$\begin{pmatrix} n-1 & -1 & \dots & -1 \\ -1 & n-1 & & -1 \\ & & \ddots & \\ -1 & & & n-1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} n \\ -n \\ 0 \\ \vdots \\ 0 \end{pmatrix} = n \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\text{Let } x_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad x_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \dots \quad x_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ -1 \end{pmatrix}$$

Then  $x_1, \dots, x_{n-1}$  are  $n-1$  indep  
eigenvector with value  $n$ .

$$\text{Thus } \lambda(K_n) = \{0, n, \dots, n\}$$

# Path Embedding

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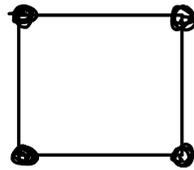
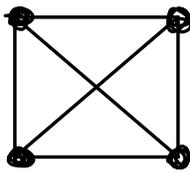
Def  $G$  (guest) &  $H$  (host)

$$G = (V, E) \text{ \& } H = (V, E)$$

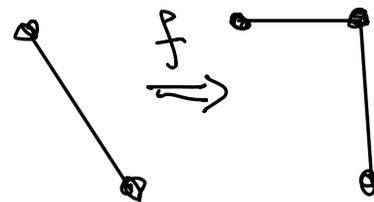
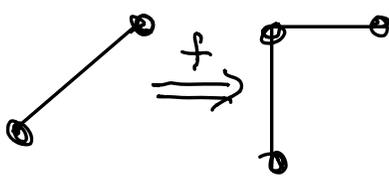
Path embedding:  $f: E_G \rightarrow \text{Paths in } H$

$$f((u, v)) \equiv \text{path from } u \text{ to } v \text{ in } H$$

eg  $G \equiv K_4$  &  $H \equiv C_4$



if  $e \in E_G \cap E_H$  then  $f(e) = e$



$$\text{Congestion} \equiv \max_{e \in H} |\{P \mid e \in P\}|$$

$$\text{Dilation} \equiv \max_P \{|P|\}$$

$$\text{Cong} = 3$$

$$\text{Dil} = 2$$

## Laplacian a sum of laplacians

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$$E_{ij} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad P_{ij} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Note  $L_G = \sum_{(i,j) \in E} w_{ij} E_{ij}$

pf. Write product  $BCB^T$  as sum of outer products.

Def  $A$  is diagonally dominate if

$$A_{ii} \geq \sum_{i \neq j} |A_{ij}|$$

Claim  $A$  is SDD then

$$A = \sum_{\substack{i \neq j \\ A_{ij} < 0}} -A_{ij} E_{ij} + \sum_{\substack{A_{ij} > 0 \\ i \neq j}} A_{ij} P_{ij} + \text{Diagonal} \\ D \geq 0$$

Thm  $f: G \rightarrow H$  is a path embedding  
 with congestion  $c$  & dilation  $d$   
 then  $L_G \leq c \cdot d L_H$

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pf Let  $G = L(G)$  &  $H = L(H)$

Let  $E_1, \dots, E_m$  edge Laplacians of  $G$ .  
 Let  $P_1, \dots, P_m$  path Laplacians of  $H$

Note  $\sum P_i \leq c H$

$$\frac{x^T G x}{x^T H x} \leq \frac{x^T \sum E_i x}{(1/c) x^T \sum P_i x} = \frac{c \sum x^T E_i x}{\sum x^T P_i x}$$

$$\leq c \max_i \frac{x^T E_i x}{x^T P_i x} \leq c \cdot d$$

Thus  $x^T G x \leq c \cdot d x^T H x$

$\Rightarrow G \leq c d H$

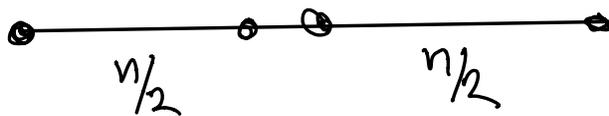
## Back to $\lambda_2$ for $P_n$

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Let  $f: K_n \xrightarrow{\text{path}} P_n$  (unique)

Congestion

$$C = (n/2)^2$$



dilation  $\equiv n-1$

By Thm  $\lambda_2(K_n) \leq C \cdot d \lambda_2(P_n)$

$$\frac{n}{C \cdot d} \leq \lambda_2(P_n)$$

$$\frac{n}{(n/2)^2 (n-1)} = \frac{4}{n(n-1)} \leq \lambda_2(P_n)$$

Thus  $\frac{4}{n^2} \leq \lambda_2 \leq \frac{12}{n^2}$