

The Basic Iterative Method 15-859N 3/8/21

Goal: Solve $Au=b$ approximately

Simpler prob^o: solve $a \cdot x = 1$ approximately
without division?

Def $\bar{a} = 1 - a$ or $a = 1 - \bar{a}$

thus

$$\frac{1}{a} = \frac{1}{1-\bar{a}} = 1 + \bar{a} + \bar{a}^2 + \dots \quad \text{for } |\bar{a}| < 1 \\ \text{or } 0 < a < 2.$$

$$\text{Let } u^{(k)} = 1 + \bar{a} + \dots + \bar{a}^{k-1}$$

$$\text{Recurrence } u^{(k+1)} = 1 + \bar{a}(u^{(k)})$$

Suppose goal: $a \cdot x = b$

$$\text{Set } v^{(k)} = b u^{(k)} \quad \text{or} \quad u^{(k)} = v^{(k)} / b$$

recurrence:

$$\begin{aligned} v^{(k+1)} &= b \left(1 + \frac{\bar{a}}{b} v^{(k)} \right) = b + \bar{a} v^{(k)} \\ &= b + (1-a) v^{(k)} = v^{(k)} + (b - a v^{(k)}) \\ 0 &< a < 2 \end{aligned}$$

Error Analysis

$$\text{Set } \varepsilon^{(k)} = \frac{b}{a} - v^{(k)} \quad \varepsilon^{(0)} = \frac{b}{a}$$

$$\begin{aligned} \varepsilon^{(k)} &= b(1 + \bar{a} + \bar{a}^2 + \dots) - b(1 + \bar{a} + \dots + \bar{a}^{k-1}) \\ &= b\bar{a}^k (1 + \bar{a} + \dots) = \frac{b\bar{a}^k}{a} = \bar{a}^k \left(\frac{b}{a} \right) \\ &= \bar{a}^k \varepsilon^{(0)} \end{aligned}$$

converges iff $|\bar{a}| < 1$

Richardson's Method

view 1 $u^{(m+1)} = u^{(m)} + (b - Au^{(m)})$

$\uparrow$
 residual error

view 2 $u^{(m+1)} = (I - A)u^{(m)} + b$

$u^{(m+1)} = Gu^{(m)} + b \quad G = (I - A)$

Good: 1) $O(w(A))$ work per step.
(parallel)

2) A fixed point is a solution.

Bad: 1) May not converge.

2) May do so very slowly.

The error for Richardson

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Suppose $A\bar{u}=b$ & $G=(I-A)$

Def Error $\xi^{(m)} = u^{(m)} - \bar{u}$

Claim $\xi^{(m+1)} = G \xi^{(m)}$

$$\begin{aligned} \text{pt } G \xi^{(m)} &= G(u^{(m)} - \bar{u}) = Gu^{(m)} - (I-A)\bar{u} \\ &\approx Gu^{(m)} - \bar{u} + A\bar{u} \\ &= Gu^{(m)} - \bar{u} + b \\ &= (Gu^{(m)} + b) - \bar{u} \\ &= u^{(m+1)} - \bar{u} = \xi^{(m+1)} \end{aligned}$$

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Note: To converge $\lim_{m \rightarrow \infty} G^m \xi^{(0)} = 0$

to converge $\forall \xi^{(0)}$ then $\lim_{m \rightarrow \infty} G^m = 0$

Note $A \text{ sym} \Rightarrow G \text{ sym}$

thus $G = \sum \lambda_i (v_i v_i^T)$ $\lambda_i \in \lambda(G)$
 v_i eigenvector

and converge iff $\forall i \ |\lambda_i| < 1$

In general $\exists i$ s.t. $|\lambda_i| \geq 1$

Simple modification
 to get convergence!

The Extrapolated Method

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A sym

Goal Get Richardson to converge.

Note $Ax=b$ iff $\gamma Ax = \gamma b$ $\gamma \neq 0$

New recurrence

$$u^{(k+1)} = u^{(k)} - \gamma(b - Ax)$$

or
$$u^{(k+1)} = (I - \gamma A)u^{(k)} + \gamma b$$

set
$$G_\gamma = (I - \gamma A)$$

Goal:
$$\min_{\gamma} \rho(G_\gamma) = \max_i |\lambda_i(G_\gamma)|$$

Set $M = \lambda_{\max}(A)$ & $m = \lambda_{\min}(A)$

Claim $\gamma = \frac{2}{M+m}$ is optimal

Note $Ax = \lambda_A x$ iff $G_\gamma x = (1 - \gamma \lambda_A) x$

Simple fact: Let $Q(z)$ be a polynomial
then $\lambda_i \in \lambda(G)$ iff $Q(\lambda_i) \in \lambda(Q(G))$

Here $Q(z) = 1 - \gamma z$

$$\lambda_{\max}(G) \leq (1 - \gamma \lambda_A) = 1 - \gamma m = 1 - \frac{2m}{M+m} = \frac{M-m}{M+m}$$

$$\begin{aligned} \lambda_{\min}(G) &\geq 1 - \gamma M = 1 - \frac{2M}{M+m} = \frac{m-M}{M+m} \\ &= -\frac{M-m}{M+m} \end{aligned}$$

$$\text{thus } \nabla(G_\gamma) \leq \frac{M-m}{M+m}$$

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Def condition number of $A \equiv \frac{M_A}{m_A} = \kappa(A)$

$$\nabla(G_8) = \frac{M-m}{M+m} = \frac{\frac{M}{m} - \frac{m}{m}}{\frac{M}{m} + \frac{m}{m}} = \frac{\kappa(A)-1}{\kappa(A)+1}$$

Set $\kappa = \kappa(A)$

$$\nabla = 1 - \frac{2}{\kappa+1} \approx 1 - \frac{1}{\kappa}$$

After κ iteration $E_r = (1 - \frac{1}{\kappa})^\kappa \approx \frac{1}{e}$

We need $\approx \kappa$ iteration/bit.

e.g. $A = L(P_n) \quad \kappa(A) \approx n^2$

Need $\approx n^2$ iteration/bit.

Polynomial Acceleration

Assume A SPD & $A\bar{x} = b$

$$x^{(m+1)} = Gx^{(m)} + b \quad G = (I - A)$$

$$\varepsilon^{(m)} = G^m \varepsilon^{(0)}$$

Idea: Use all previous $x^{(i)}$

i.e. Compute $x^{(0)}, \dots, x^{(n)}$

Pick $\alpha_i \in \mathbb{R}$ s.t. $\sum \alpha_i = 1$

return $u^{(n)} = \sum \alpha_i x^{(i)}$

Question: How to pick the α_i ?

A different view!

$$\begin{aligned}
 \underline{\text{Def}} \quad \tilde{\mathbf{x}}^{(n)} &= \mathbf{u}^{(n)} - \bar{X} \\
 &= \sum_{i=0}^{n-1} \alpha_i \chi^{(i)} - \bar{X} \quad \sum_{i=0}^{n-1} \alpha_i = 1 \\
 &= \sum \alpha_i (\chi^{(i)} - \bar{X}) \\
 &= \sum \alpha_i \mathbf{x}^{(i)} = \sum \alpha_i G^i \mathbf{x}^{(0)} \\
 &= \left(\sum \alpha_i G^i \right) \mathbf{x}^{(0)} = Q_n(G) \mathbf{x}^{(0)}
 \end{aligned}$$

where $Q_n(z) = \alpha_0 + \dots + \alpha_{n-1} z^{n-1}$

$$\sum \alpha_i = 1$$

Goal: For each n pick $Q_n(z)$

$$\min \sigma(Q_n(G))$$

Note: $\lambda(Q_n(G)) = Q_n(\lambda(G))$

Chebyshev Polys & Acceleration

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Def Chebyshev Polynomials

$$T_0(w) = 1 \quad T_1(w) = w$$

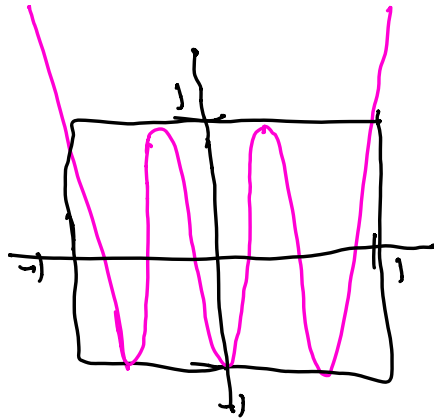
$$T_{n+1}(w) = 2w T_n(w) - T_{n-1}(w)$$

$$\text{eg } T_2(w) = 2w \cdot w - 1 = 2w^2 - 1$$

$$T_3(w) = 2w(2w^2 - 1) - w =$$

$$= 4w^3 - 2w - w = 4w^3 - 3w$$

$$\text{eg } T_6(w) =$$



Some Simple Facts:

$$1) T_n(1) = 1 \Rightarrow \sum \text{coef} = 1$$

$$2) T_n(-1) = (-1)^n$$

$$3) \begin{aligned} n \text{ odd} &\Rightarrow T_n(w) \text{ is odd} \\ n \text{ even} &\Rightarrow T_n(w) \text{ is even} \end{aligned}$$

Harder Facts

$$1) T_n(w) = \frac{1}{2} \left[(w + \sqrt{w^2 - 1})^n + (w - \sqrt{w^2 - 1})^n \right]$$

(HW)

$$2) \text{Unique Poly Satisfying}$$

$$T_n(\cos(\theta)) = \cos(n\theta)$$

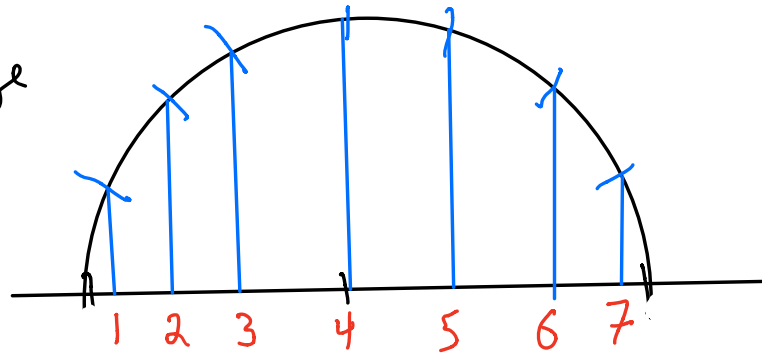
$$3) \text{For } |x| \leq 1$$

$$T_n(x) = \cos(n \arccos(x))$$

$$4) \text{Roots } x_k = \cos\left(\frac{k}{n+1}\pi\right)$$

$\mathbb{L}_{\text{image}}$

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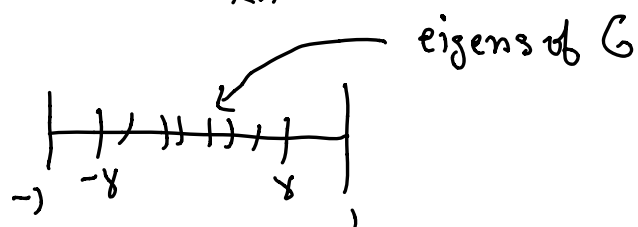
Thm $d > 1$ Let $H_n(w) = T_n(w) / T_n(1/d)$

$$1) \max_{-1 \leq w \leq 1} |H_n(w)| = 1 / T_n(1/d)$$

2) $H_n(w)$ is min such poly

Chebyshev Plus Extrapolated Method

Let $\kappa = \kappa(A)$ & $\gamma = \frac{\kappa-1}{\kappa+1}$

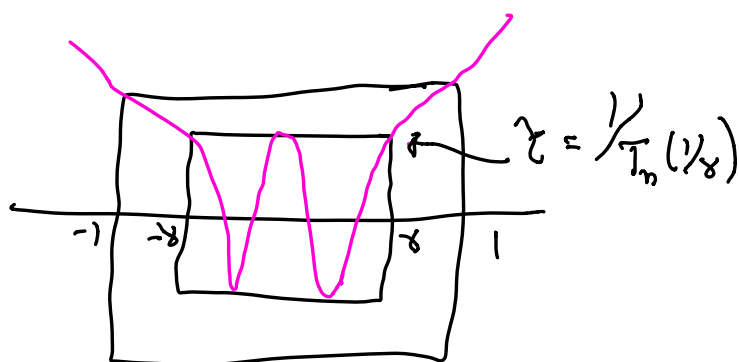


We will use $\tilde{T}_n = T_n(x/\gamma) / T_n(1/\gamma)$

for our poly acceleration!

Note $\tilde{T}_n(1) = 1$ thus $\sum \text{coef} = 1$

picture



Note: $|\lambda(G)| \leq \gamma$ thus $|\lambda(Q_n(G))| < \gamma$

Goal bnd from below $T_n(1/\gamma)$

$$1/\gamma = \frac{\kappa+1}{\kappa-1} = 1 + \frac{2}{\kappa-1} = 1 + 2/\mu \quad \text{where } \mu = \frac{1}{\kappa-1}$$

Note $T_n(x) = \frac{1}{2} \left[(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n \right]$
 $\geq \frac{1}{2} (x + \sqrt{x^2 - 1})^n \text{ for } x \geq 1$

$$\begin{aligned} T_n(1+2\mu) &\geq \frac{1}{2} \left(1+2\mu + \sqrt{(1+2\mu)^2 - 1} \right)^n \\ &= \frac{1}{2} \left(1+2\mu + 2\sqrt{\mu(\mu+1)} \right)^n \\ &= \frac{1}{2} \left(\sqrt{\mu} + \sqrt{\mu+1} \right)^{2n} \quad (*) \end{aligned}$$

note

$$\left(\sqrt{\mu} + \sqrt{\mu+1} \right)^2 = \left(\frac{1}{\sqrt{k-1}} + \frac{\sqrt{k}}{\sqrt{k-1}} \right)^2 = \frac{(1+\sqrt{k})^2}{k-1} = \frac{\sqrt{k}+1}{\sqrt{k}-1}$$

thus $(*) = \frac{1}{2} \left(\frac{\sqrt{k}+1}{\sqrt{k}-1} \right)^n$

Finally: $T_n(1/8) \geq \frac{1}{2} \left(\frac{\sqrt{k}+1}{\sqrt{k}-1} \right)^n$

or: $r \leq 2 \left(\frac{\sqrt{k}-1}{\sqrt{k}+1} \right)^n$

Thm Convergence Rate for Extrapolated
Chebyshev is $\Theta(1/\sqrt{k})$

Back to P_n $K(L(P_n)) \approx n^2$

$\therefore$ Convergence for Cheb is $\Theta(1/n)$

n -iteration per bit

Optimal for multi-term recurrence

since we are only doing "local" ops!

From Polynomial Recurrence to Iterative Alg

Consider a poly defined by

$$Q_0(x) = 1$$

$$Q_1(x) = \alpha_1 x + \beta_1 \quad \alpha_1 + \beta_1 = 1$$

$$Q_{n+1}(x) = \alpha_n x Q_n(x) + \beta_n Q_{n-1}(x) \quad \alpha_n + \beta_n = 1$$

Iterative Alg

$$u^{(n+1)} = \alpha_n (G u^{(n)} + b) + \beta_n u^{(n-1)}$$

to show $\varepsilon^{(n)} = u^{(n)} - \bar{u}$ then $\varepsilon^{(n)} = Q_n(G) \varepsilon^{(0)}$

$$Q_{n+1}(G) \varepsilon^{(0)} = [\alpha_n G Q_n(G) + \beta_n Q_{n-1}(G)] \varepsilon^{(0)} \quad \text{induct!}$$

$$= \alpha_n G \underbrace{Q_n(G) \varepsilon^{(0)}}_{\varepsilon^{(n)}} + \beta_n Q_{n-1}(G) \varepsilon^{(0)}$$

$$= \alpha_n G(\varepsilon^{(n)}) + \beta_n (\varepsilon^{(n-1)})$$

$$= \alpha_n G(u^{(n)} - \bar{u}) + \beta_n (u^{(n-1)} - \bar{u})$$

$$= \alpha_n (Gu^{(n)} - \bar{u} + b) + \beta_n u^{(n-1)} - \beta_n \bar{u}$$

$$= \alpha_n (Gu^{(n)} + b) + \beta_n u^{(n-1)} - \bar{u}$$

$$= u^{(n+1)} - \bar{u} = \varepsilon^{(n+1)}$$