15-859N: Spectral Graph Theory and The Laplacian Paradigm **Spring 2020**

Lecture 9: Bounding Eigenvalues , 2020: Feb 5

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9.1 Overview

In this lecture, we introduce techniques to bound eigenvalues of graph adjacency and Laplacian matrix, we also present a simple but powerful technique for lower bounding eigenvalues, called *path embedding argument*.

9.2 Mediant of Fractions

We first prove a simple but useful claim, many of our later theorems and lemmas rely on it.

Definition 9.1. Let $\left\{\frac{a_i}{b_i}\right\}_{i\in[n]}$ be the set of n *fractions*, where for all $1 \le i \le n$, $a_i, b_i > 0$.

Definition 9.2. The *mediant* of $\left\{\frac{a_i}{b_i}\right\}_{i\in[n]}$ is defined as

$$\frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i}$$

Claim 9.3. *Let $\left\{\frac{a_i}{b_i}\right\}_{i\in[n]}$ be n fractions, then*

$$\min_{1\le j\le n}\frac{a_j}{b_j} \le \frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i} \le \max_{1\le k\le n}\frac{a_k}{b_k}$$

Proof: Intuitively, if we view (b_i, a_i) as points in $\mathbb{R}^2$, then $\frac{a_i}{b_i}$ is the slope of the line that goes through origin and (b_i, a_i). Suppose we take an average of all points x-coordinates and y-coordinates, then the resulting point should have slope in between of the largest slope and smallest slope. We formalize such intuition below. We prove the second inequality by induction on number of fractions. For $n = 1$, it holds trivially. Suppose for $n = k - 1$ it holds, consider k points,then without loss of generality, assume $\frac{a_k}{b_k}$ maximizes the fraction, let $\frac{a_j}{b_j}$ be the maximum fraction for the first $k - 1$ terms, then by induction hypothesis, we know

$$
\begin{aligned}
\frac{\sum_{i=1}^{k-1} a_i}{\sum_{i=1}^{k-1} b_i} &\le \frac{a_j}{b_j} \\
&\le \frac{a_k}{b_k} \\
\frac{\sum_{i=1}^{k-1} a_i}{a_k} &\le \frac{\sum_{i=1}^{k-1} b_i}{b_k} \\
1 + \frac{\sum_{i=1}^{k-1} a_i}{a_k} &\le 1 + \frac{\sum_{i=1}^{k-1} b_i}{b_k} \\
\frac{\sum_{i=1}^{k} a_i}{a_k} &\le \frac{\sum_{i=1}^{k} b_i}{b_k} \\
\frac{\sum_{i=1}^{k} a_i}{\sum_{i=1}^{k} b_i} &\le \frac{a_k}{b_k}
\end{aligned}
$$

The proof for the first inequality is analogous. ■

9.3 Courant-Fisher's Theorem and its applications

Here we will prove Courant-Fisher's Theorem, and use it to obtain many bounds on eigenvalues of adjacency and Laplacian matrix.

Theorem 9.4 (Courant-Fisher)**.** *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix with eigenvalues $\mu_1 \ge \mu_2 \ge \ldots \ge \mu_n$, then*

$$
\begin{aligned}
\mu_k &= \max_{S\subseteq\mathbb{R}^n,\dim(S)=k}\left(\min_{x\in S}\frac{x^TAx}{x^Tx}\right) \\
&= \min_{T\subseteq\mathbb{R}^n,\dim(T)=n-k+1}\left(\max_{x\in T}\frac{x^TAx}{x^Tx}\right)
\end{aligned}
$$

Proof: One can view CF Theorem as a two person game.We will prove for the max-min case. Let $x_1, \ldots, x_n$ be eigenvectors of A, corresponding to eigenvalues $\mu_1, \ldots, \mu_n$.
($\mu_k \le$ max-min) : We specify a strategy for two players. Player 1 first picks $S = \operatorname{span}(x_1, \ldots, x_k)$ and player 2 picks arbitrary $x \in S$, we can write $x = \sum_{i=1}^{k} c_i x_i$, so

$$
\begin{aligned}
\frac{x^TAx}{x^Tx} &= \frac{x^T\sum_{i=1}^{k} c_i A x_i}{x^Tx} \\
&= \frac{x^T\sum_{i=1}^{k} c_i \mu_i x_i}{x^Tx} \\
&= \frac{\sum_{i=1}^{k} c_i^2\mu_i x_i^T x_i + \sum_{i\ne j} c_i c_j \mu_j x_i^T x_j}{\sum_{i=1}^{k} c_i^2 x_i^T x_i + \sum_{i\ne j} c_i c_j x_i^T x_j} \\
&= \frac{\sum_{i=1}^{k} c_i^2\mu_i}{\sum_{i=1}^{k} c_i^2} \\
&\ge \frac{c_k^2\mu_k}{c_k^2} \qquad \text{by mediant of fraction} \\
&= \mu_k
\end{aligned}
$$

($\mu_k \geq$ max-min) : For any $S \subseteq \mathbb{R}^n$ with $\dim(S) = k$, we wish to show $\min_{x \in S} \frac{x^T Ax}{x^T x} \leq \mu_k$. Consider $T = \operatorname{span}(x_k, \ldots, x_n)$, notice $\dim(T) = n - k + 1$, so $T \cap S \neq \{0\}$, i.e., there intersection is non-trivial, so

$$\min_{x \in S} \frac{x^T Ax}{x^T x} \leq \min_{x \in S \cap T} \frac{x^T Ax}{x^T x}$$

Since $x \in T$, we can write it as $x = \sum_{i=k}^n c_i x_i$, so

$$\begin{aligned}
\frac{x^T Ax}{x^T x} &= \frac{\sum_{i=k}^n c_i^2 \mu_i}{\sum_{i=k}^n c_i^2} \\
&\leq \frac{c_k^2 \mu_k}{c_k^2} && \text{by mediant of fraction} \\
&= \mu_k
\end{aligned}$$

■

With Courant-Fisher in hand, we are ready to give some bounds on eigenvalues of adjacency and Laplacian matrix.

Lemma 9.5. *Let G be a connected graph and A be its adjacency matrix, and d_{avg} denote the average degree, and $d_{\max}$ denote the max degree, and let $\mu_1 \geq \ldots \geq \mu_n \in \lambda(A)$, then*

$$d_{\text{avg}} \leq \mu_1 \leq d_{\max}$$

Proof: $d_{\text{avg}} \leq \mu_1$: By Courant-Fisher, we know that

$$\begin{aligned}
\mu_1 &= \max_{x \neq 0} \frac{x^T Ax}{x^T x} \\
&\geq \frac{\mathbf{1}^T A \mathbf{1}}{\mathbf{1}^T \mathbf{1}} \\
&= \frac{\mathbf{1}^T \begin{bmatrix} d(v_1) \\ d(v_2) \\ \vdots \\ d(v_n) \end{bmatrix}}{n} \\
&= \frac{\sum_{i=1}^n d(v_i)}{n} \\
&= d_{\text{avg}}
\end{aligned}$$

$\mu_1 \leq d_{\text{maxz}}$: Let x be an eigenvector of A such that $Ax = \mu_1 x$, since G is connected, by Perron-Frobenious Theorem, we can make x be a positive vector, i.e., $x > \mathbf{0}$. Without loss of generality, assume $x_v \geq x_u$, for all $u \in V$. Observe that

$$(Ax)_v = \mu_1 x_v$$

So

$$\begin{aligned}\mu_1 &= \frac{(Ax)_v}{x_v} \\ &= \frac{\sum_{\{u,v\}\in E} A_{v,u}x_u}{x_v} \\ &\le \frac{\sum_{\{u,v\}\in E} A_{v,u}x_v}{x_v} \\ &= \sum_{\{u,v\}\in E} A_{v,u} \\ &= d(v) \\ &\le d_{\max}\end{aligned}$$

■

Using this lemma, we can obtain a tighter bound of μ_1 for unit-weight d-regular graph.

Lemma 9.6. *Let G be a connected unit-weight graph, A be its adjacency matrix and μ_1 be the largest eigenvalue of A, then G is d-regular if and only if $\mu_1 = d_{max} = d$.*

Proof: $(\Rightarrow)$: Suppose G is d-regular, then $d_{\text{avg}} = d_{\max} = d$, by 9.5, $d \le \mu_1 \le d \to d = \mu_1$.
$(\Leftarrow)$: Suppose $\mu_1 = d_{\max}$, let $Ax = \mu_1 x$, and without loss of generality, assume $x > \mathbf{0}$, $x_v \ge x_u$. Then

$$\begin{aligned}(Ax)_v &= d_{\max}x_v \\ \sum_{\{u,v\}\in E} x_u &= d_{\max}x_v\end{aligned}$$

Clearly, $d(v) \le d_{\max}$, on the other hand, $\sum_{\{u,v\}\in E} \frac{x_u}{x_v} = d_{\max} \le d(v)$, combine these two we get $d(v) = d_{\max}$, so $x_v = 1$ and all nodes u incident to v also have $x_u = 1$. Propagate this argument using connectivity of G to all nodes, we get

$$x = \begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}$$

and $Ax = dx$ indicates that sum of each row of A is d, and G is d-regular. ■

Lemma 9.7. *Let G be a graph and L be its Laplacian with eigenvalues $0 \le \lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$, then $\lambda_n \le 2d_{\max}$.*

Proof: Recall that $L = D - A$, so $\lambda_n \le \lambda_{\max}(D) - \lambda_{\min}(A)$, notice $\lambda_{\max}(D)$ is $d_{\max}$, and by Perron-Frobenius Theorem, we know $|\lambda_{\min}(A)| \le \lambda_{\max}(A)$, so $-\lambda_{\min}(A) \le \lambda_{\max}(A)$, and we have

$$\begin{aligned}\lambda_n &\le \lambda_{\max}(D) - \lambda_{\min}(A) \\ &\le \lambda_{\max}(D) + \lambda_{\max}(A) \\ &\le d_{\max} + d_{\max} \\ &= 2d_{\max}\end{aligned}$$

■

We conclude this section by proving a theorem related to Loewner order. Recall that for two symmetric matrices A, B, we have $A \preceq B$ if for all $x \in \mathbb{R}^n$, $x^T Ax \le x^T Bx$.

Theorem 9.8. *Let $A, B \in \mathbb{R}^{n\times n}$ be two symmetric matrices, if $A \preceq B$, then for all $1 \le k \le n$,*

$$\lambda_k(A) \le \lambda_k(B)$$

where $\lambda_k(\cdot)$ is the k^{th} largest eigenvalue of a matrix.

Proof: We use Courant-Fisher Theorem. Let $T \subseteq \mathbb{R}^n$ be a subspace such that

$$\lambda_k(B) = \min_{\dim(T)=n-k+1} \max_{x\in T} \frac{x^T Bx}{x^T x}$$

Since $A \preceq B$, $\frac{x^T Ax}{x^T x} \le \frac{x^T Bx}{x^T x}$, this means

$$\lambda_k(B) \ge \max_{x\in T} \frac{x^T Ax}{x^T x}$$

Notice such subspace T might not be the minimum that minimizes $\lambda_k(A)$, so we have

$$\lambda_k(B) \ge \max_{x\in T} \frac{x^T Ax}{x^T x} \ge \lambda_k(A)$$

■

9.4 Bounding eigenvalues for Laplacian

Recall that in 9.7, we have already shown that the largest eigenvalue of Laplacian is upper bounded by $2d_{\max}$, since Laplacian is positive semi-definite and has rank $n-1$, we can write its eigenvalue set as

$$\lambda(L_G) = 0 \le \lambda_1 \le \lambda_2 \le \ldots \le \lambda_n \le 2d_{\max}$$

We know try to study more carefully about λ_2, since knowing λ_2 is critical in many applications. We start with some simple graphs, namely, the path graph.

Definition 9.9. Let P_n be the unit-weight path graph on n edges and $n+1$ nodes.

By Courant-Fisher, we know that

$$\lambda_2 = \min_{v^T\mathbf{1}=0} \frac{v^T Lv}{v^T v}$$

We will try to throw out some vectors that are orthogonal to $\mathbf{1}$, in order to get upper bounds on λ_2.

Consider $x = \begin{bmatrix} -n/2 \\ -n/2+1 \\ \vdots \\ n/2-1 \\ n/2 \end{bmatrix}$, observe that $x^T\mathbf{1} = 0$ since sum over all its entries is 0. Moreover,

$$\begin{aligned} x^T L(P_n)x &= \sum_{i=1}^{n} (x_{i+1} - x_i)^2 \\ &= n \end{aligned}$$

since in general, we can write $x^T L x = \sum_{\{i,j\}\in E} C_{i,j}\left(x_i - x_j\right)^2$, where $C_{i,j}$ is the conductance of edge $\{i,j\}$, on unit-weight path graph, conductance of each edge is simply 1. And

$$x^T x = \sum_{i=1}^{n+1} x_i^2 = 2\sum_{i=1}^{n/2} i^2 \approx \frac{2}{3}\left(n/2\right)^3 = n^3/12$$

So the Rayleigh quotient gives

$$\frac{x^T L\left(P_n\right) x}{x^T x} \approx \frac{n}{n^3/12} = \frac{12}{n^2}$$

Now let's try to obtain some lower bounds on λ_2. Before doing so, let's first understand eigenvalues of some even "simpler" graph, namely, the complete graph on n nodes, K_n. It's not hard to see that

$$L\left(K_n\right) = \begin{bmatrix} n-1 & -1 & -1 & \dots & -1 \\ -1 & n-1 & -1 & \dots & -1 \\ & & \ddots & & \\ -1 & -1 & -1 & \dots & n-1 \end{bmatrix}$$

So

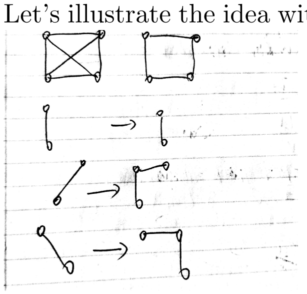
$$L\left(K_n\right)\begin{bmatrix}1\\-1\\0\\\vdots\\0\end{bmatrix} = \begin{bmatrix}n\\-n\\0\\\vdots\\0\end{bmatrix} = n\begin{bmatrix}1\\-1\\0\\\vdots\\0\end{bmatrix}$$

which means $\begin{bmatrix}1\\-1\\0\\\vdots\\0\end{bmatrix}$ is an eigenvector of $L\left(K_n\right)$ with eigenvalue n. One can move entries $1, -1$ of the vector all the way down to obtain $n-1$ orthogonal eigenvectors, each has eigenvalue n, so $\lambda\left(L\left(K_n\right)\right) = \{0, n\}$. We will make use of such easy eigenvalues in our latter argument.

Now we are ready to introduce a powerful technique called *path embedding*.

Definition 9.10. Let $G = (V,E), H = (V,E')$ be two unit-weight graphs, where G is called the *guest*, and H is called the *host*. A *path embedding from G to H* is an embedding $f : E_G \hookrightarrow$ paths of H.

Let's illustrate the idea with an example. Consider $G = K_4, H = C_4$, then the embedding is given as follows:



where edges are the cycle are embedded to themselves, and two diagonal edges are embedded as above.

Definition 9.11. Let f be a path embedding, the *congestion* is defined as

$$\max_{e\in H} \left|\{P : P \text{ is a path in } \mathrm{im}\,(f), e \in P\}\right|$$

and the *dilation* is defined as

$$\max_{P \text{ is a path in im}(f)} \{|P|\}$$

In above example, the congestion is 3, the top edge has been used by 3 paths, and dilation is 2.

Theorem 9.12. *Let G, H be unit-weight graphs, and f be a path embedding from G to H with congestion c and dilation d, then*

$$L(G) \preceq cd \cdot L(H)$$

Before proving the theorem, we introduce some definitions on subgraph Laplacian.

Definition 9.13. Let e_i be an edge of G, and u, v be its two end points, let $\chi_{u,v}$ be the indicator vector such that its u^{th} entry is 1 and v^{th} entry is -1, then define the *edge Laplacian* as $E_i = \chi_{u,v}\chi_{u,v}^T$.
The *path Laplacian* is defined similarly: we fill in entries corresponding to the path its standard values in Laplacian, and all other entries 0.

Proof: Let $E_1, \ldots, E_m$ be edge Laplacians of G, and $P_1, \ldots, P_m$ be path Laplacians of H with respect to path embedding f. The first observation is

$$\sum_{i=1}^{m} E_i = L(G)$$

Another observation is congestion c captures the maximum multiplicity of an edge in a path embedding, so

$$\sum_{i=1}^{m} P_i \le c \cdot H$$

one can view $c \cdot H$ as duplicating c edges for every single edge in H, and each path consumes 1 copy for each edge on it. By this pointwise inequality, we easily obtain that

$$\left(\sum_{i=1}^{m} P_i\right) x \le c \cdot Hx$$
$$x^T \left(\sum_{i=1}^{m} P_i\right) x \le x^T c \cdot Hx$$
$$\sum_{i=1}^{m} P_i \preceq c \cdot H$$

Now pick x to be orthogonal to $\mathbf{1}$, and consider

$$\begin{aligned}
\frac{x^T Gx}{x^T Hx} &\le \frac{x^T \left(\sum_{i=1}^m E_i\right) x}{\frac{1}{c} x^T \left(\sum_{i=1}^m P_i\right) x} \\
&= \frac{c x^T \left(\sum_{i=1}^m E_i\right) x}{x^T \left(\sum_{i=1}^m P_i\right) x} \\
&= \frac{c \sum_{i=1}^m x^T E_i x}{\sum_{i=1}^m x^T P_i x} \\
&\le c \max_{1 \le i \le n} \frac{x^T E_i x}{x^T P_i x} && \text{by mediant of fraction} \\
&\le cd
\end{aligned}$$

Notice the last inequality is equivalent to say

$$x^T E_i x \le d x^T P_i x$$

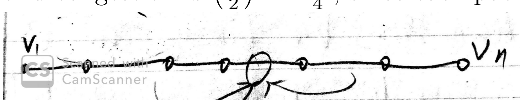
To see this, suppose $e_i = \{v_1, v_{d+1}\}$, so LHS is $(x_1 - x_{d+1})^2$, and RHS is $d\sum_{i=1}^{d}(x_i - x_{i+1})^2$. To minimize RHS, we can set each consecutive x_i's to have the same difference, so its

$$d\sum_{i=1}^{d}\left(\frac{x_{d+1}-x_1}{d}\right)^2 = d^2\frac{(x_{d+1}-x_1)^2}{d^2} = (x_{d+1}-x_1)^2$$

Thus, we have for all $x \perp \mathbf{1}$,

$$x^T L(G) x \le x^T (cd \cdot L(H)) x \Rightarrow L(G) \preceq cd \cdot L(H)$$

■ Consider the path embedding f from K_n to P_n, notice it's unique. Dilation for this embedding is $n-1$, and congestion is $\left(\frac{n}{2}\right)^2 = \frac{n^2}{4}$, since each path uses the edge on middle. A picture to illustrate this:



Apply 9.12 and 9.8, we have

$$\begin{aligned}
\lambda_2(L(K_n)) &\le cd\lambda_2(L(P_n)) \\
&= (n-1)\frac{n^2}{4}\lambda_2(L(P_n)) \\
\Rightarrow \lambda_2(P_n) &\ge \frac{n}{(n-1)n^2/4} \\
&\ge \frac{4}{n^2}
\end{aligned}$$

As a result, we get an upper and lower bound on λ_2 of P_n:

$$\frac{4}{n^2} \le \lambda_2 \le \frac{12}{n^2}$$