GOLDEN-THOMPSON INEQUALITY

For $n \times n$ complex matrices, the matrix exponential is defined by Taylor series as

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

For commuting matrices A and B we see that $e^{A+B} = e^A e^B$ by multiplying the Taylor series. This identity is not true for general non-commuting matrices. In fact, it always fails if A and B do not commute, see [2].

Theorem 1 (Golden-Thompson Inequality)**.** *For arbitrary self-ajoint matrices A and B, one has*

$$\operatorname{tr}(e^{A+B}) \leq \operatorname{tr}(e^A e^B).$$

For a survey of Golden-Thompson and other trace inequalities, see [2]. In the present note, we give a proof of Golden-Thompson inequality following [1] Theorem 9.3.7.

Remarks. 1. Golden-Thompson inequality holds for arbitrary unitary-invariant norm replacing the trace, see [1] Theorem 9.3.7.

2. A version of Golden-Thompson inequality for three matrices fails: $\operatorname{tr}(e^{A+B+C}) \not\leq \operatorname{tr}(e^A e^B e^C)$.

The proof of Golden-Thompson inequality is based on Lie Product Formula:

Theorem 2 (Lie Product Formula)**.** *For arbitrary matrices A and B, we have*

$$e^{A+B} = \lim_{N \to \infty} (e^{A/N} e^{B/N})^N.$$

Proof. We first compare

$$X_N = e^{(A+B)/N} \quad \text{and} \quad Y_N = e^{A/N} e^{B/N}.$$

As $N \to \infty$, Taylor's expansion gives

$$\begin{aligned} X_N &= 1 + \frac{A+B}{N} + O(N^{-2}), \\ Y_N &= \left[1 + \frac{A}{N} + O(N^{-2})\right] \left[1 + \frac{B}{N} + O(N^{-2})\right] \\ &= 1 + \frac{A}{N} + \frac{B}{N} + O(N^{-2}). \end{aligned}$$

This shows that

$$X_N - Y_N = O(N^{-2}). \tag{1}$$

Now, to compare $X_N^N - Y_N^N$, we shall use the following bound:

Claim. For arbitrary matrices X and Y, we have

$$\|X^N - Y^N\| \leq NM^{N-1}\|X - Y\|,$$

where $M = \max(\|X\|, \|Y\|)$.

To prove this claim, we write the telescoping sum

$$\begin{aligned} X^N - Y^N &= (X^N - X^{N-1}Y) + (X^{N-1}Y + X^{N-2}Y^2) + \cdots + (XY^{N-1} - Y^N) \\ &= X^{N-1}(X - Y) + X^{N-2}(X - Y)Y + \cdots + (X - Y)Y^{N-1}. \end{aligned}$$

Each of the N terms in this sum is bounded by $M^{N-1}\|X - Y\|$. This proves Claim.

To complete the proof of Lie Product Formula, we shall use Claim for $X = X_N$, $Y = Y_N$. Since

$$\begin{aligned} \|X_N\| &\leq e^{\|A+B\|/N} \leq e^{(\|A\|+\|B\|)/N}, \\ \|Y_N\| &\leq \|e^{A/N}e^{B/N}\| \leq e^{(\|A\|+\|B\|)/N}, \end{aligned}$$

we have

$$M^N = \max(\|X\|, \|Y\|)^N \leq e^{\|A\|+\|B\|}.$$

Therefore, using Claim and the bound (1), we conclude that

$$\|X_N - Y_N\| \leq Ne^{\|A\|+\|B\|}O(N^{-2}) = O(1/N).$$

This completes the proof of Lie Product Formula. □

Another ingredient we will need is the following.

Proposition 3. *For arbitrary matrix X and a positive integer m, one has*

$$|\operatorname{tr}(X^m)| \leq \operatorname{tr}(|X|^m).$$

In the right hand side, we use the notation $|X| = (X^*X)^{1/2}$.

This proposition is a straightforward consequence of Weyl's Majorant Theorem, which states eigenvalues of a matrix are dominated by the singular values:

Theorem 4 (Weyl's Majorant Theorem)**.** *Let A be an $n \times n$ matrix with singular values $s_1 \geq \cdots \geq s_n$ and eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ arranged so that $|\lambda_1| \geq \cdots \geq |\lambda_n|$. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that $f(e^t)$ is convex and increasing in t. Then*

$$\sum_{i=1}^n f(|\lambda_i|) \leq \sum_{i=1}^n f(s_i).$$

For a proof, see [1] Theorem 2.3.6.

Proposition 3 follows from Weyl's Majorant Theorem for the function $f(x) = x^m$:

$$|\operatorname{tr}(X^m)| = \Big|\sum_{i=1}^{n} \lambda_i^m\Big| \le \sum_{i=1}^{n} |\lambda_i|^m \le \sum_{i=1}^{n} s_i^m = \operatorname{tr}(|X|^m).$$

Proof of Golden-Thompson Inequality. Fix a natural number N and consider

$$X = e^{A/2^N}, \quad X = e^{B/2^N}.$$

To prove Golden-Thompson Inequality, it suffices to show that

$$\operatorname{tr}((XY)^{2^N}) \le \operatorname{tr}(X^{2^N} Y^{2^N}). \tag{2}$$

Indeed, if (2) holds then, taking limit as $N \to \infty$ we see that the left hand side of (2) converges to $\operatorname{tr}(e^{A+B})$ by Lie Product Formula, while the right hand side equals $\operatorname{tr}(e^A e^B)$.

To prove (2), we use Proposition 3 and note that $|XY|^2 = (XY)^*(XY) = YX^2Y$. We thus have

$$\operatorname{tr}(XY)^{2^N} \le \operatorname{tr}(YX^2Y)^{2^{N-1}} = \operatorname{tr}(X^2Y^2)^{2^{N-1}},$$

where the last equality follows from the trace property $\operatorname{tr}(UV) = \operatorname{tr}(VU)$.

Continuing this procedure for X^2 and Y^2, we obtain

$$\operatorname{tr}(X^2Y^2)^{2^{N-1}} \le \operatorname{tr}(X^4Y^4)^{2^{N-2}}.$$

After N steps, we arrive at the bound (2). This proves Golden-Thompson Inequality. $\square$

References

[1] R. Bhatia, *Matrix analysis*. Graduate Texts in Mathematics, 169. Springer-Verlag, New York, 1997. xii+347 pp.

[2] , D. Petz, *A survey of trace inequalities*, in Functional Analysis and Operator Theory, 287-298, Banach Center Publications, 30 (Warszawa 1994). Online at `http://www.renyi.hu/~petz/pdf/64.pdf`