15-859N Spectral Graph Theory and The Laplacian Paradigm, Fall 2018

Homework 3 **Due: Friday Dec 7**

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Instructions. Collaboration is permitted in groups of size at most three. You must write the names of your collaborators on your solutions and must write your solutions alone.

Question	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

(25) 1. **Resistance Theorem**

In this problem we show that if we have a graph G with two attachment vertices a and b and we only have attachment to these two vertices then we can replace the entire graph with a single edge from a to b with resistance R_{ab} in G.

Prove the following theorem.

Theorem 0.1. *Let a and b be two vertices of a graph G with Laplacian L. Let R_{ab} the effective resistance in G from a to b and H_{ab} the Laplacian of the unit weight single edge graph from a to b, i.e., $H_{ab} = \chi_{ab}\chi_{ab}^T$. Then for all $x \in \Re^n$,*

$$x^T H_{ab} x \leq R_{ab} \cdot x^T L x \tag{1}$$

and there exist an x such that for all α, the inequality holds with equality for $x + \alpha\bar{1}$.

Hint: First prove the theorem for $x = L^+ w$. At some point in your proof you will need to use Cauchy-Schwartz.

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(25) 2. **Leverage Scores and Resistors**

Recall that much of this class has focused on the theorems regrading effective resistance of graphs. The goal of this problem is to determine if we can generalized these theorems to arbitrary matrices. Let B be the edge by vertex matrix of a connected graph G with a diagonal conductance matrix C then the Laplacian of G is B^TCB. If b_i is the ith row of B corresponding to the ith edge e_i of G then the effective resistance from one end of e_i to the other is $b_i(B^TCB)^\dagger b_i^T$. If we define $\bar{B} = C^{1/2}B$ then the resistance is $b_i(\bar{B}^T\bar{B})^\dagger b_i^T$. This motivates to following definition.

Let $A^{m\times n}$ be a matrix of rank n.

Definition 1. *The* **Leverage Score** $\sigma(A, a)$ *where a is a column vector of size n is $a^T(A^TA)^\dagger a$ where $\dagger$ is the pseudoinverse. If a_i is the ith row of A the $\sigma_i(A) = \sigma(A, a_i^T)$.*

The goal of this problem is to determine what if any of the properties of effective resistance carry over the leverage scores.

1. Show that the leverage score of a nonzero row vector with itself is one.
2. Show that the column space $Col(A)$ of A and the left null space $Null_L(A)$ of A form an orthogonal bases of R^m. We will think of these vectors in $f \in \mathbb{R}^m$ as the flows. In the case when $A = \sqrt{C}B$ what kind of flows are the $Col(A)$ and the $Null_L(A)$?
3. We next prove a generalization of Foster's Theorem. Show that the sum of the row leverage scores of A is $rank(A)$. In this problem assume that A is not of full rank.
4. We next prove a generalization of Thomsons Principle. Suppose that x is a solution to the system $A^TAx = b$ were b is in the column space of A^T. Show that the flow $f = Ax$ is the unique minimum energy flow such that $A^Tf = b$. We define the energy of f to be f^Tf.
5. We next prove a generalization of Rayleigh's Monotonicity Law. If we increase a row of A by scaling it by $1 + c$ for $c > 0$ or add a new row then no leverage score except the changed one will increase.
6. We next prove a generalization of Spielman-Srivastava Graph Sparsification Theorem. We say that $A^TA \approx_\epsilon B^TB$ if $(1-\epsilon)A^TA \preccurlyeq B^TB \preccurlyeq (1+\epsilon)A^TA$. Prove that that there exist a matrix QB where B is a subset of m' rows of A and Q is a nonnegative diagonal matrix where $m' = cn \log n$ for some constant c and $A^TA \approx_\epsilon B^TB$.
7. Prove a variant of the fact that conductors add when placed in parallel. In particular prove a relationship between $\sigma(a, a)$, $\sigma(A, a)$, and $\sigma(\bar{A}, a)$ where $\bar{A}$ is the matrix A with row a appended.

   Hint: Consider the Sherman-Morrison formula.

   Can you find a more general formula?

Research questions:

1. We also showed that the effective resistance forms a metric over the **vertices**. Thus in our generalization we should be looking for a metric on the columns of A. Suppose we define the score between two columns as $D_{ij} = \chi_{ij}^T(A^TA)^{-1}\chi_{ij}$ where

χ_{ij} is the column vector with a 1 and -1 at i and j respectively. Does our score say anything interesting about the relationship of two columns? Is D_{ij} a metric on the columns of A.

2. Is there theory of random walks for leverage scores, either on the clumns or rows of A?
3. If there in such a theory as randoms walks does commute time make sense and is related to leverage score?

(25) 3. **Linear Sized Sparsifier**

In this problem we prove a step that helps us show that every Laplacian (graph) has linear-sized $1+\epsilon$ sparsifiers. You can assume the following lemma is true.

Lemma 1. *Given vectors $v_1, v_2, ..., v_m \in \mathbb{R}^n$ with*

$$\sum_{i \leq m} v_i v_i^T = I_n$$

then there exists scalars $s_i \geq 0$ with $|\{i : s_i \neq 0\}| \leq 4n$ so that

$$I_n \preceq \sum_{i \leq m} s_i v_i v_i^T \preceq 9 I_n$$

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With the help of the above lemma, prove that for every weighted graph G there exists a subgraph $H \subseteq G$ with at most $4n$ edges such that $L_G \preceq L_H \preceq 9L_G$.

(25) 4. **Loewner Inequalities**

1. Using the path embedding argument prove that for any spanning tree T of a connected unweighted graph G,

$$L_T \preceq L_G \preceq str_T(G) L_T$$

where $str_T(G)$ is the stretch of G in T.

2. Let A and B be symmetric positive definite n by n matrices.
Show that $A \preceq \alpha B$ if and only if $\lambda_{\max}(B^{-1}A) \leq \alpha$
3. Let $G = (V, E, w)$ and $H = (V, E', w')$ be two weighted graphs. Lets define the stretch between of G in H as

$$str_H(G) = \sum_{e \in E} w_G(e) \cdot ER_H(e)$$

Using Part 2 prove that for any subgraph H of a connected weighted graph G,

$$L_H \preceq L_G \preceq str_H(G) L_H$$

where $str_H(G)$ is the stretch of G in H.

4. Let that $G = (V, E, w)$ and $H = (V', E', w')$ be two weighted graphs such that $W \subset V \cap V'$.

   Definition 2. *Let x vary over column vectors corresponding to the vertices in W, y vary over column vectors corresponding to the vertices $V \setminus W$, and y' vary over column vectors corresponding to the vertices $V' \setminus W$. We define the Lowener inequality $L_G \preceq_W L_H$ if for all x:*

   $$\min_y \begin{pmatrix} x \\ y \end{pmatrix}^T L_G \begin{pmatrix} x \\ y \end{pmatrix} \leq \min_{y'} \begin{pmatrix} x \\ y' \end{pmatrix}^T L_H \begin{pmatrix} x \\ y' \end{pmatrix}$$

   (a) Let $G = (V, E, w)$ be a weighted graph over the vertices V. Suppose that A is the Schur complement of L_G after pivoting out some single vertex/variable, say, v_n from L_G. Show that A is the Laplacian of a graph, say H on the vertices $V_1, \ldots, V_{n-1}$.

   (b) Furthermore, suppose we pivoted out from $G = (V, E, w)$ all the vertices $v \in W \subset V$. Denote the residual graph with $H = (V', E', w')$ where $V' = V \setminus W$. Show that:

   $$L_H \preceq_{V'} L_G \preceq_{V'} L_H$$