15-859N Spectral Graph Theory and The Laplacian Paradigm, Fall 2018

Homework 3

Due: Friday Dec 7
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Instructions. Collaboration is permitted in groups of size at most three. You must write the names of your collaborators on your solutions and must write your solutions alone.

Question | Points | Score |
---|---|---|
1 | 25 | |
2 | 25 | |
3 | 25 | |
4 | 25 | |
Total: 100 |

(25) 1. **Resistance Theorem**

In this problem we show that if we have a graph $G$ with two attachment vertices $a$ and $b$ and we only have attachment to these two vertices then we can replace the entire graph with a single edge from $a$ to $b$ with resistance $R_{ab}$ in $G$.

Prove the following theorem.

**Theorem 0.1.** Let $a$ and $b$ be two vertices of a graph $G$ with Laplacian $L$. Let $R_{ab}$ the effective resistance in $G$ from $a$ to $b$ and $H_{ab}$ the Laplacian of the unit weight single edge graph from $a$ to $b$, i.e., $H_{ab} = \chi_{ab}\chi_{ab}^T$. Then for all $x \in \mathbb{R}^n$,

$$x^T H_{ab} x \leq R_{ab} \cdot x^T L x$$

and there exist an $x$ such that for all $\alpha$, the inequality holds with equality for $x + \alpha I$.

Hint: First prove the theorem for $x = L^+ w$. At some point in your proof you will need to use Cauchy-Schwartz.

(25) 2. **Leverage Scores and Resistors**

Recall that much of this class has focused on the theorems regrading effective resistance of graphs. The goal of this problem is to determine if we can generalized these theorems
to arbitrary matrices. Let $B$ be the edge by vertex matrix of a connected graph $G$ with a diagonal conductance matrix $C$ then the Laplacian of $G$ is $B^T C B$. If $b_i$ is the $i$th row of $B$ corresponding to the $i$th edge $e_i$ of $G$ then the effective resistance from one end of $e_i$ to the other is $b_i (B^T C B)^{1/2} b_i^T$. If we define $B = C^{1/2} B$ then the resistance is $b_i (B^T B)^{1/2} b_i^T$. This motivates to following definition.

Let $A^{m \times n}$ be a matrix of rank $n$.

**Definition 1.** The Leverage Score $\sigma(A, a)$ where $a$ is a column vector of size $n$ is $a^T (A^T A)^{1/2} a$ where $\dagger$ is the pseudoinverse. If $a_i$ is the $i$th row of $A$ the $\sigma_i(A) = \sigma(A,a_i^T)$.

The goal of this problem is to determine what if any of the properties of effective resistance carry over the leverage scores.

1. Show that the leverage score of a nonzero row vector with itself is one.
2. Show that the column space $Col(A)$ of $A$ and the left null space $Null_L(A)$ of $A$ form an orthogonal bases of $\mathbb{R}^m$. We will think of these vectors in $f \in \mathbb{R}^m$ as the flows.
3. If we increase a row of $A$ by scaling it by $1 + c$ for $c > 0$ or add a new row then no leverage score except the changed one will increase.
4. We next prove the following: Suppose that $x$ is a solution to the system $A^T A x = b$ where $b$ is in the column space of $A^T$. Show that the flow $f = A x$ is the unique minimum energy flow such that $A^T f = b$. We define the energy of $f$ to be $f^T f$.
5. We next prove a generalization of Rayleigh’s Monotonicity Law. If we increase a row of $A$ by scaling it by $1 + c$ for $c > 0$ or add a new row then no leverage score except the changed one will increase.

Research questions:

1. We also showed that the effective resistance forms a metric over the vertices. Thus in our generalization we should be looking for a metric on the columns of $A$. Suppose we define the score between two columns as $D_{ij} = \chi_{ij}^T (A^T A)^{-1} \chi_{ij}$ where
\( \chi_{ij} \) is the column vector with a 1 and \(-1\) at \( i \) and \( j \) respectively. Does our score say anything interesting about the relationship of two columns? Is \( D_{ij} \) a metric on the columns of \( A \).

2. Is there theory of random walks for leverage scores, either on the columns or rows of \( A \)?

3. If there in such a theory as random walks does commute time make sense and is related to leverage score?

(25) \[ 3. \textbf{Linear Sized Sparsifier} \]

In this problem we prove a step that helps us show that every Laplacian (graph) has linear-sized \( 1 + \epsilon \) sparsifiers. You can assume the following lemma is true.

**Lemma 1.** Given vectors \( v_1, v_2, \ldots, v_m \in \mathbb{R}^n \) with

\[
\sum_{i \leq m} v_i v_i^T = I_n
\]

then there exists scalars \( s_i \geq 0 \) with \( |\{i : s_i \neq 0\}| \leq 4n \) so that

\[
I_n \preceq \sum_{i \leq m} s_i v_i v_i^T \preceq 9I_n
\]

With the help of the above lemma, prove that for every weighted graph \( G \) there exists a subgraph \( H \subseteq G \) with at most \( 4n \) edges such that \( L_G \preceq L_H \preceq 9L_G \).

(25) \[ 4. \textbf{Loewner Inequalities} \]

1. Using the path embedding argument prove that for any spanning tree \( T \) of a connected unweighted graph \( G \),

\[
L_T \preceq L_G \preceq \text{str}_T(G)L_T
\]

where \( \text{str}_T(G) \) is the stretch of \( G \) in \( T \).

2. Let \( A \) and \( B \) be symmetric positive definite \( n \times n \) matrices. Show that \( A \preceq \alpha B \) if and only if \( \lambda_{\text{max}}(B^{-1}A) \leq \alpha \)

3. Let \( G = (V, E, w) \) and \( H = (V, E', w') \) be two weighted graphs. Let define the stretch between of \( G \) in \( H \) as

\[
\text{str}_H(G) = \sum_{e \in E} w_G(e) \cdot ER_H(e)
\]

Using Part 2 prove that for any subgraph \( H \) of a connected weighted graph \( G \),

\[
L_H \preceq L_G \preceq \text{str}_H(G)L_H
\]

where \( \text{str}_H(G) \) is the stretch of \( G \) in \( H \).
4. Let that \( G = (V, E, w) \) and \( H = (V', E', w') \) be two weighted graphs such that \( W \subset V \cap V' \).

**Definition 2.** Let \( x \) vary over column vectors corresponding to the vertices in \( W \), \( y \) vary over column vectors corresponding to the vertices \( V \setminus W \), and \( y' \) vary over column vectors corresponding to the vertices \( V' \setminus W \). We define the Lowener inequality \( L_G \preceq_W L_H \) if for all \( x \):

\[
\min_y \begin{pmatrix} x \\ y \end{pmatrix}^T L_G \begin{pmatrix} x \\ y \end{pmatrix} \leq \min_{y'} \begin{pmatrix} x \\ y' \end{pmatrix}^T L_H \begin{pmatrix} x \\ y' \end{pmatrix}
\]

(a) Let \( G = (V, E, w) \) be a weighted graph over the vertices \( V \). Suppose that \( A \) is the Schur complement of \( L_G \) after pivoting out some single vertex/variable, say, \( v_n \) from \( L_G \). Show that \( A \) is the Laplacian of a graph, say \( H \) on the vertices \( V_1, \ldots, V_{n-1} \).

(b) Furthermore, suppose we pivoted out from \( G = (V, E, w) \) all the vertices \( v \in W \subset V \). Denote the residual graph with \( H = (V', E', w') \) where \( V' = V \setminus W \). Show that:

\[
L_H \preceq_V L_G \preceq V \ L_H
\]