15-859N Spectral Graph Theory and The Laplacian Paradigm, Fall 2018

Homework 1 Version 1.1 **Due: Wednesday Oct 3**

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Instructions. Collaboration is permitted in groups of size at most three. You must write the names of your collaborators on your solutions and must write your solutions alone.

Question	Points	Score
1	25	
2	25	
Total:	50	

(25) 1. **Pseudoinverses**

Wikipedia defines the Moore-Penrose inverse or Pseudoinverse to be the operation that takes an $n \times m$ matrix A and returns an $m \times n$ A^+ satisfying the following constraints. We will assume that A is real this will simpify the constraints to:

1. $AA^+A = A$
2. $A^+AA^+ = A^+$
3. $(AA^+)^T = AA^+$
4. $(A^+A)^T = A^+A$

Recall that in class we defined the pseudoinverse of a symmetric real matrix M to be $U\Lambda^+U^T$ where $M = U\Lambda U^T$ is the spectral decomposition of M.

1. Show that our definition agrees with the Moore-Penrose inverse and it is the only solution.
2. In the case where A is real symmetric do we need all four axioms or constraints?

(25) 2. **Effective Resistance and Selfloops**

1 ER from the inverse of Laplacian

Given a connected graph of conductors $G = (V, E, c)$ (for simplicity assume $V = [n]$), let L be its Laplacian. Fix a destination $n \in V$, and let L_n be obatined from L by deleting the last row and last column.

1. Prove that L_n has full rank.
2. Prove that $(L_n^{-1})_{i,i}$ is equal to the effective resistance between i and n.

2 How do self-loops affect hitting times?

Given a connected graph $G = (V, E, c)$ (for simplicity assume $V = [n]$), let L be its Laplacian. Fix a destination $n \in V$. We defined the hitting time h_i to be the expected number of steps we need to take from i to n. For any $i \neq n$, we have $h_i = 1 + \sum_j \frac{c_{i,j}}{c_i} h_j$ with $h_n = 0$, which can be modeled written as $L_n h = [c_1, ..., c_n]^T$ where L_n is obtained from L by deleting the last row and column.

1. Suppose that we add a self-loop of positive weight at some vertices of G to make G'. Show that the hitting time is still a solution of $L_n h = [c'_1, ..., c'_{n-1}]^T$ where $c'_i \geq c_i$ for every i. Note that L_n is still obtained from the Laplacian of G.
2. Intuition tells that adding a self-loop should increase the hitting time of every vertex. Prove it formally by showing that if h, h' satisfy $L_n h = c$ and $L_n h' = c'$ where $c \leq c'$ (which means that $c_i \leq c'_i$ for all i), $h \leq h'$.
3. Does the above fact also holds when L_n is replaced by any other positive-definite matrix? In other words, if A is a symmetric, positive definite matrix (so it is invertible), $b \leq b'$, and $Ax = b, Ax' = b'$, is it true that $x \leq x'$?