GOLDEN-THOMPSON INEQUALITY

For $n \times n$ complex matrices, the matrix exponential is defined by Taylor series as
\[
e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.
\]
For commuting matrices $A$ and $B$ we see that $e^{A+B} = e^A e^B$ by multiplying the Taylor series. This identity is not true for general non-commuting matrices. In fact, it always fails if $A$ and $B$ do not commute, see [2].

**Theorem 1** (Golden-Thompson Inequality). For arbitrary self-adjoint matrices $A$ and $B$, one has
\[
\text{tr}(e^{A+B}) \leq \text{tr}(e^A e^B).
\]

For a survey of Golden-Thompson and other trace inequalities, see [2]. In the present note, we give a proof of Golden-Thompson inequality following [1] Theorem 9.3.7.

**Remarks.**
2. A version of Golden-Thompson inequality for three matrices fails:
\[
\text{tr}(e^{A+B+C}) \not\leq \text{tr}(e^A e^B e^C).
\]
The proof of Golden-Thompson inequality is based on Lie Product Formula:

**Theorem 2** (Lie Product Formula). For arbitrary matrices $A$ and $B$, we have
\[
e^{A+B} = \lim_{N \to \infty} (e^{A/N} e^{B/N})^N.
\]

**Proof.** We first compare
\[
X_N = e^{(A+B)/N} \quad \text{and} \quad Y_N = e^{A/N} e^{B/N}.
\]
As $N \to \infty$, Taylor's expansion gives
\[
X_N = 1 + \frac{A+B}{N} + O(N^{-2}),
\]
\[
Y_N = \left[ 1 + \frac{A}{N} + O(N^{-2}) \right] \left[ 1 + \frac{B}{N} + O(N^{-2}) \right]
= 1 + \frac{A}{N} + \frac{B}{N} + O(N^{-2}).
\]
This shows that
\[
X_N - Y_N = O(N^{-2}).
\]
Now, to compare $X_N^N - Y_N^N$, we shall use the following bound:

**Claim.** For arbitrary matrices $X$ and $Y$, we have

$$\|X^N - Y^N\| \leq NM^{N-1}\|X - Y\|,$$

where $M = \max(\|X\|, \|Y\|)$.

To prove this claim, we write the telescoping sum

$$X^N - Y^N = (X^N - X^N - 1Y) + (X^N - 1Y + X^{N-2}Y^2) + \cdots + (XY^{N-1} - Y^N)$$

$$= X^{N-1}(X - Y) + X^{N-2}(X - Y)Y + \cdots + (X - Y)Y^{N-1}.$$

Each of the $N$ terms in this sum is bounded by $M^{N-1}\|X - Y\|$. This proves Claim.

To complete the proof of Lie Product Formula, we shall use Claim for $X = X_N$, $Y = Y_N$. Since

$$\|X_N\| \leq e^{\|A+B\|}/N \leq e^{(\|A\|+\|B\|)/N},$$

$$\|Y_N\| \leq \|e^{A/N}e^{B/N}\| \leq e^{(\|A\|+\|B\|)/N},$$

we have

$$M^N = \max(\|X\|, \|Y\|)^N \leq e^{\|A\|+\|B\|}.$$

Therefore, using Claim and the bound (1), we conclude that

$$\|X_N - Y_N\| \leq Ne^{\|A\|+\|B\|}O(N^{-2}) = O(1/N).$$

This completes the proof of Lie Product Formula. \hfill \Box

Another ingredient we will need is the following.

**Proposition 3.** For arbitrary matrix $X$ and a positive integer $m$, one has

$$|\text{tr}(X^m)| \leq \text{tr}(|X|^m).$$

In the right hand side, we use the notation $|X| = (X^*X)^{1/2}$.

This proposition is a straightforward consequence of Weyl’s Majorant Theorem, which states eigenvalues of a matrix are dominated by the singular values:

**Theorem 4** (Weyl’s Majorant Theorem). Let $A$ be an $n \times n$ matrix with singular values $s_1 \geq \cdots \geq s_n$ and eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ arranged so that $|\lambda_1| \geq \cdots \geq |\lambda_n|$. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that $f(e^t)$ is convex and increasing in $t$. Then

$$\sum_{i=1}^n f(|\lambda_i|) \leq \sum_{i=1}^n f(s_i).$$
For a proof, see [1] Theorem 2.3.6.

Proposition 3 follows from Weyl’s Majorant Theorem for the function $f(x) = x^m$.

$$|\text{tr}(X^m)| = \left|\sum_{i=1}^{n} \lambda_i^m\right| \leq \sum_{i=1}^{n} |\lambda_i|^m \leq \sum_{i=1}^{n} s_i^m = \text{tr}(|X|^m).$$

**Proof of Golden-Thompson Inequality.** Fix a natural number $N$ and consider

$$X = e^{A/2N}, \quad X = e^{B/2N}.$$  

To prove Golden-Thompson Inequality, it suffices to show that

$$\text{(2)} \quad \text{tr}((XY)^{2N}) \leq \text{tr}(X^{2N}Y^{2N}).$$

Indeed, if (2) holds then, taking limit as $N \to \infty$ we see that the left hand side of (2) converges to $\text{tr}(e^{A+B})$ by Lie Product Formula, while the right hand side equals $\text{tr}(e^{A}e^{B})$.

To prove (2), we use Proposition 3 and note that $|XY|^2 = (XY)^*(XY) = YX^2Y$. We thus have

$$\text{tr}(XY)^{2N} \leq \text{tr}(YX^2Y)^{2N-1} = \text{tr}(X^{2N}Y^{2N-1}),$$

where the last equality follows from the trace property $\text{tr}(UV) = \text{tr}(VU)$.

Continuing this procedure for $X^2$ and $Y^2$, we obtain

$$\text{tr}(X^2Y^2)^{2N-1} \leq \text{tr}(X^4Y^4)^{2N-2}.$$

After $N$ steps, we arrive at the bound (2). This proves Golden-Thompson Inequality.

**References**
