

## Lecture 8

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In this lecture, we continue the proof of Cheeger's inequality and explore similar bounds on the largest eigenvalue of the normalized Laplacian. Recall that the normalized Laplacian is given by  $\mathcal{L} = D^{-1/2}L_G D^{-1/2}$ , where

$$D^{-1/2} = \begin{pmatrix} \frac{1}{\sqrt{d(1)}} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sqrt{d(2)}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sqrt{d(n)}} \end{pmatrix},$$

and  $d(i)$  is the degree of vertex  $i$ . When  $S \subseteq V$ , we define  $\delta(S)$  as the set of edges with exactly one endpoint in  $S$ , and  $\text{vol}(S) = \sum_{i \in S} d(i)$ . The conductance of  $S$  is defined as

$$\phi(S) = \frac{|\delta(S)|}{\min(\text{vol}(S), \text{vol}(V - S))},$$

and the conductance of  $G$  is defined as  $\phi(G) = \min_{S \subseteq V} \phi(S)$ . Finally, let  $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$  denote the eigenvalues of  $\mathcal{L}$ .

## 1 Cheeger's Inequality

**Theorem 1 (Cheeger's inequality, upper bound)** *We have  $\phi(G) \leq \sqrt{2\lambda_2}$ .*

Last time, we showed that, for any vector  $y \in \mathbb{R}^n$  with  $\sum_{i \in V} d(i)y(i) = 0$ , we can find  $S_t \subseteq \text{supp}(y) = \{i \in V : y(i) \neq 0\}$  such that  $\frac{|\delta(S_t)|}{\text{vol}(S_t)} \leq \sqrt{2R(y)}$ , where

$$R(y) = \frac{\sum_{(i,j) \in E} (y(i) - y(j))^2}{\sum_{i \in V} d(i)y(i)^2}.$$

We also saw that  $\lambda_2 = \min R(y)$ . The issue is that we may have  $\text{vol}(S_t) > \text{vol}(V - S_t)$ . To fix this, we will modify  $y$  so that  $\text{vol}(\text{supp}(y)) \leq m$  (recall that  $\text{vol}(V) = 2m$ ).

The idea is to pick  $c$  such that the two sets  $\{i : y(i) < c\}$  and  $\{i : y(i) > c\}$  both have volume at most  $m$ , then find  $S_t$  for both of them and take the best one.

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<sup>0</sup>This lecture is derived from Lau's 2012 notes, Week 2, <http://appsrv.cse.cuhk.edu.hk/~chi/csc5160/notes/L02.pdf> and Lau's 2015 notes, Lecture 4, <https://cs.uwaterloo.ca/~lapchi/cs798/notes/L04.pdf>.

**Claim 2** Let  $z = y - ce$ , where  $e \in \mathbb{R}^n$  is the vector of all ones. Then

$$(i) \quad z^\top Dz \geq y^\top Dy.$$

$$(ii) \quad z^\top L_G z = y^\top L_G y.$$

(iii) Let  $z_+(i) = \max(0, z(i))$  and  $z_-(i) = \min(0, z(i))$ . Then  $\min(R(z_+), R(z_-)) \leq R(z) \leq R(y)$  and  $\text{supp}(z_+), \text{supp}(z_-)$  both have volume at most  $m$ .

Given the claim, we can finish the proof of Cheeger's inequality. Using the algorithm from last lecture, we find  $S_+ \subseteq \text{supp}(z_+)$ ,  $S_- \subseteq \text{supp}(z_-)$  with

$$\begin{aligned} \min(\phi(S_+), \phi(S_-)) &= \min\left(\frac{|\delta(S_+)|}{\text{vol}(S_+)}, \frac{|\delta(S_-)|}{\text{vol}(S_-)}\right) \leq \min(\sqrt{2R(z_+)}, \sqrt{2R(z_-)}) \\ &\leq \sqrt{2R(y)}, \end{aligned}$$

so that  $\phi(G) \leq \min(\phi(S_+), \phi(S_-)) \leq \min \sqrt{2R(y)} = \sqrt{2\lambda_2}$ , as desired.

**Proof of claim:**

(i) Let  $f(c) = (y - ce)^\top D(y - ce) = \sum_{i \in V} d(i)(y(i) - c)^2$ .

We have  $f'(c) = \sum_{i \in V} (-2y(i)d(i) + 2cd(i)) = 2c \sum_{i \in V} d(i)$ , by  $\sum_i y(i)d(i) = 0$ . Also,  $f''(c) = 2 \sum_i d(i) > 0$ , so that  $f$  is minimized when  $f'(c) = 0 \iff c = 0$ , so that  $z^\top Dz \geq y^\top Dy$ , as desired.

(ii) Indeed,

$$\begin{aligned} z^\top L_G z &= \sum_{(i,j) \in E} (z(i) - z(j))^2 = \sum_{(i,j) \in E} ((y(i) - c) - (y(j) - c))^2 \\ &= \sum_{(i,j) \in E} (y(i) - y(j))^2 = y^\top L_G y. \end{aligned}$$

(iii) Note that

$$z^\top Dz = \sum_{i \in V} d(i)z(i)^2 = \sum_{i \in V} d(i)z_+(i)^2 + \sum_{i \in V} d(i)z_-(i)^2 = z_+^\top D z_+ + z_-^\top D z_-,$$

and

$$z^\top L_G z \geq z_+^\top L_G z_+ + z_-^\top L_G z_-,$$

if we can show that  $(z(i) - z(j))^2 \geq (z_+(i) - z_+(j))^2 + (z_-(i) - z_-(j))^2$  for all  $i, j$ . This follows since if  $z(i)$  and  $z(j)$  have the same sign, then clearly  $(z(i) - z(j))^2 = (z_+(i) - z_+(j))^2 + (z_-(i) - z_-(j))^2$  (where one of the two terms is zero), while if  $z(i)$  and  $z(j)$  have opposite signs then

$$\begin{aligned} (z(i) - z(j))^2 &= z(i)^2 - 2z(i)z(j) + z(j)^2 \\ &\geq z(i)^2 + z(j)^2 \\ &\geq (z_+(i) - z_+(j))^2 + (z_-(i) - z_-(j))^2, \end{aligned}$$

since  $-2z(i)z(j)$  is positive in this case. Therefore,

$$R(y) = \frac{y^\top L_G y}{y^\top D y} \geq R(z) = \frac{z^\top L_G z}{z^\top D z} \geq \frac{z_+^\top L_G z_+ + z_-^\top L_G z_-}{z_+^\top D z_+ + z_-^\top D z_-} \geq \min(R(z_+), R(z_-)),$$

and from our choice of  $c$ , we have  $\text{vol}(z_+) \leq m$  and  $\text{vol}(z_-) \leq m$ .  $\square$

Renato Paes Leme and David Applegate observe that the cuts generated by considering the vectors  $z_+$  and  $z_-$  correspond to sweep cuts in the original vector  $y$ , and so the overall analysis giving the upper bound on  $\phi(G)$  can be thought of as analyzing the sweep cuts of  $y$ .

## 2 Bounds on largest eigenvalue

In the last lecture, we proved that  $\lambda_n \leq 2$ . Note that

$$\lambda_n = \max_{x \in \mathbb{R}^n} \frac{x^\top \mathcal{L} x}{x^\top x} = \max_{x \in \mathbb{R}^n} \frac{x^\top D^{-1/2} L_G D^{-1/2} x}{x^\top x} = \max_{y \in \mathbb{R}^n} \frac{y^\top L_G y}{y^\top D y},$$

where we take  $y = D^{-1/2} x$ . We also claim the following

**Claim 3**  $\lambda_n = 2$  if and only if  $G$  has a bipartite component.

We can easily show the if direction. If  $G$  has a bipartite component  $S$  with sides  $L, R$ , define a vector  $y \in \mathbb{R}^n$  as  $y(i) = 1$  if  $i \in L$ ,  $y(i) = -1$  if  $i \in R$  and  $y(i) = 0$  otherwise.

If  $\delta(A, B)$  denotes the set of edges with one endpoint in  $A$  and another in  $B$ , we have

$$\frac{y^\top L_G y}{y^\top D y} = \frac{\sum_{(i,j) \in E} (y(i) - y(j))^2}{\sum_{i \in V} d(i) y(i)^2} = \frac{4\delta(L, R)}{\text{vol}(S)} = 2.$$

Now we'll show a statement stronger than the converse:  $G$  has a bipartite component when  $\lambda_n = 2$ , and has an “almost” bipartite component when  $\lambda_n$  is close to 2. To make this more precise, consider the quantity

$$\beta(G) = \min_{\substack{S \subseteq V \\ S = L \cup R \\ L \cap R = \emptyset}} \frac{2|E(L)| + 2|E(R)| + |\delta(S)|}{\text{vol}(S)},$$

where  $E(X)$  denotes the set of edges with both endpoints in  $X$ . Alternatively,

$$\beta(G) = \min_{y \in \{-1, 0, 1\}^n} \frac{\sum_{(i,j) \in E} |y(i) + y(j)|}{\sum_{i \in V} d(i) |y(i)|},$$

where  $L = \{i : y(i) = 1\}$ ,  $R = \{i : y(i) = -1\}$  and  $S = L \cup R$ .

Since  $\lambda_n$  is the largest eigenvalue of  $\mathcal{L}$ ,  $\beta_n = 2 - \lambda_n$  is the smallest eigenvalue of  $2I - \mathcal{L} = 2I - (I - \mathcal{A}) = I + \mathcal{A}$ . Hence

$$\beta_n = \min_{x \in \mathbb{R}^n} \frac{x^\top (I + \mathcal{A})x}{x^\top x} = \min_{x \in \mathbb{R}^n} \frac{x^\top D^{-1/2} (D + \mathcal{A}) D^{-1/2} x}{x^\top x} = \min_{y \in \mathbb{R}^n} \frac{y^\top (D + A)y}{y^\top D y};$$

that is,

$$\beta_n = \min_{y \in \mathbb{R}^n} \frac{\sum_{(i,j) \in E} (y(i) + y(j))^2}{\sum_{i \in V} d(i) y(i)^2}.$$

Trevisan proves the following very nice analogy to the Cheeger inequality.

**Theorem 4 (Trevisan 2009)**

$$\frac{1}{2} \beta_n \leq \beta(G) \leq \sqrt{2\beta_n}.$$

**Proof:** For the first inequality, simply note that

$$\begin{aligned} \beta_n &= \min_{y \in \mathbb{R}^n} \frac{\sum_{(i,j) \in E} (y(i) + y(j))^2}{\sum_{i \in V} d(i) y(i)^2} \leq \min_{y \in \{-1,0,1\}^n} \frac{\sum_{(i,j) \in E} (y(i) + y(j))^2}{\sum_{i \in V} d(i) y(i)^2} \\ &\leq \min_{y \in \{-1,0,1\}^n} \frac{\sum_{(i,j) \in E} 2|y(i) + y(j)|}{\sum_{i \in V} d(i) y(i)^2} = 2\beta(G), \end{aligned}$$

by noticing that  $(y(i) + y(j))^2 \leq 2|y(i) + y(j)|$  for  $y(i), y(j) \in \{-1, 0, +1\}$ .

For the second inequality, pick  $y \in \mathbb{R}^n$  satisfying  $\beta_n = \frac{y^\top (D+A)y}{y^\top y}$  and assume that  $\max_i y^2(i) = 1$  (if this is not true, scale  $y$  accordingly). Choose  $t \in [0, 1]$  uniformly at random, and set  $x(i) = 1$  if  $x(i) \geq \sqrt{t}$ ,  $x(i) = -1$  if  $x(i) \leq -\sqrt{t}$  and  $x(i) = 0$  otherwise.

**Claim 5**  $\mathbb{E}[|x(i) + x(j)|] \leq |y(i) + y(j)| \cdot (|y(i)| + |y(j)|)$  for all  $(i, j) \in E$ .

**Proof of claim:** Without loss of generality suppose  $y(i)^2 \geq y(j)^2$ . If  $y(i), y(j)$  have the same sign then

$$\begin{aligned} \mathbb{E}[|x(i) + x(j)|] &= 1 \cdot \mathbb{P}[y(j)^2 \leq t \leq y(i)^2] + 2 \cdot \mathbb{P}[t \leq y(j)^2] \\ &= y(i)^2 + y(j)^2 \\ &\leq |y(i) + y(j)| \cdot (|y(i)| + |y(j)|). \end{aligned}$$

Otherwise,  $y(i), y(j)$  have different signs, so

$$\begin{aligned} \mathbb{E}[|x(i) + x(j)|] &= 1 \cdot \mathbb{P}[y(j)^2 \leq t \leq y(i)^2] \\ &= y(i)^2 - y(j)^2 \\ &= (y(i) + y(j))(y(i) - y(j)) \leq |y(i) + y(j)| \cdot (|y(i)| + |y(j)|), \end{aligned}$$

as claimed. □

Summing over all  $(i, j) \in E$  and using Cauchy-Schwarz gives

$$\begin{aligned}
\mathbb{E} \left[ \sum_{(i,j) \in E} |x(i) + x(j)| \right] &\leq \sum_{(i,j) \in E} |y(i) + y(j)| \cdot (|y(i)| + |y(j)|) \\
&\leq \sqrt{\sum_{(i,j) \in E} (y(i) + y(j))^2} \sqrt{\sum_{(i,j) \in E} (|y(i)| + |y(j)|)^2} \\
&\leq \sqrt{\beta_n \sum_{i \in V} d(i) y(i)^2} \sqrt{\sum_{(i,j) \in E} 2(y(i)^2 + y(j)^2)} \\
&= \sqrt{2\beta_n} \sum_{i \in V} d(i) y(i)^2 \\
&= \sqrt{2\beta_n} \mathbb{E} \left[ \sum_{i \in V} d(i) |x(i)| \right],
\end{aligned}$$

so that there exists  $x \in \{-1, 0, 1\}^n$  with

$$\beta(G) \leq \frac{\sum_{(i,j) \in E} |x(i) + x(j)|}{\sum_{i \in V} d(i) |x(i)|} \leq \sqrt{2\beta_n},$$

as desired. As with the proof of the Cheeger inequality, we can find such an  $x$  easily because there are only  $n$  possible different vectors  $x$  produced by the algorithm, and these correspond to  $t = y(i)^2$  for all  $i \in V$ . □