

Eigens for Tridiagonal Sym Systems

The following method reduce prob
to Sym Tri System

Jacobi	} $n \times n$ system
Givens	
Householder	
Lanczos	} $m \times m$ $m \ll n$

$$T = \begin{pmatrix} a_1 & b_1 & & 0 \\ & b_1 & a_2 & \\ & & \ddots & b_{m-1} \\ 0 & & & b_{m-1} & a_m \\ & & & & \ddots & b_{m-1} & a_m \end{pmatrix}$$

$$P_n(\lambda) = \det(T_n - \lambda I) = \det \begin{pmatrix} a_1 - \lambda & b_1 & & \\ & b_1 & & \\ & & \ddots & \\ & & & b_{n-1} \\ & & & b_{n-1} & a_n - \lambda \end{pmatrix}$$

$$P_0 = I$$

$$P_1 = a_1 - \lambda$$

$$P_2 = (a_2 - \lambda)P_1(\lambda) - b_1^2$$

$$P_3 = (a_3 - \lambda)P_2(\lambda) - b_2^2 P_1(\lambda)$$

Claim $P_i(\lambda) = (a_i - \lambda)P_{i-1}(\lambda) - b_{i-1}^2 P_{i-2}(\lambda)$

$$P_{-1}(\lambda) = 0 \quad P_0(\lambda) = 1 \quad b_0 = 0$$

Thm If $b_1, \dots, b_{n-1} \neq 0$ then roots of P_{i-1} interlace with roots of P_i

i.e. $\lambda_1, \dots, \lambda_i$ root of P_i

$\lambda'_1, \dots, \lambda'_{i-1}$ root of P_{i-1}

$$\lambda_1 < \lambda'_1 < \lambda_2 < \dots < \lambda'_{n-1} < \lambda_n$$

Thm If $b_1 \dots b_{n-1} \neq 0$ then $\forall i \geq 0$

a) $\text{sign}(P_i(-\infty)) = +1$

b) $\text{sign}(P_i(\infty)) = -\text{sign}(P_{i+1}(\infty))$

c) $\text{Roots}(P_i)$ interlace $\text{Roots}(P_{i+1})$

pf

Case a) $i=0$ $P_0(\lambda) = 1$ done

Case b) $i=0$ $\text{sign}(P_0(\infty)) = 1$

$\text{sign}(P_1(-\infty)) = \text{sign}(a_1 - \infty) = -1$ done

Assume true $i-1$

$$P_i(\lambda) = (a_i - \lambda) P_{i-1}(\lambda) - b_{i-1}^2 P_{i-2}(\lambda)$$

a) $\lambda = -\infty$ $(+\infty)(+)$ $\left(\begin{array}{l} \text{low order} \\ \text{terms} \end{array} \right) = +\infty$ done

b) $\lambda = \infty$ $(-\infty)(\pm\infty)$ $\left(\begin{array}{l} \text{low order} \\ \text{terms} \end{array} \right) = \mp\infty$ done

c) Claim $\text{root}(P_i)$ distinct $\text{roots}(P_{i-1})$

suppose $\exists \lambda' \ P_i(\lambda') = P_{i-1}(\lambda') = 0$

$$P_i(\lambda') = (a_i - \lambda') P_{i-1}(\lambda') + b_{i-1}^2 P_{i-2}(\lambda')$$

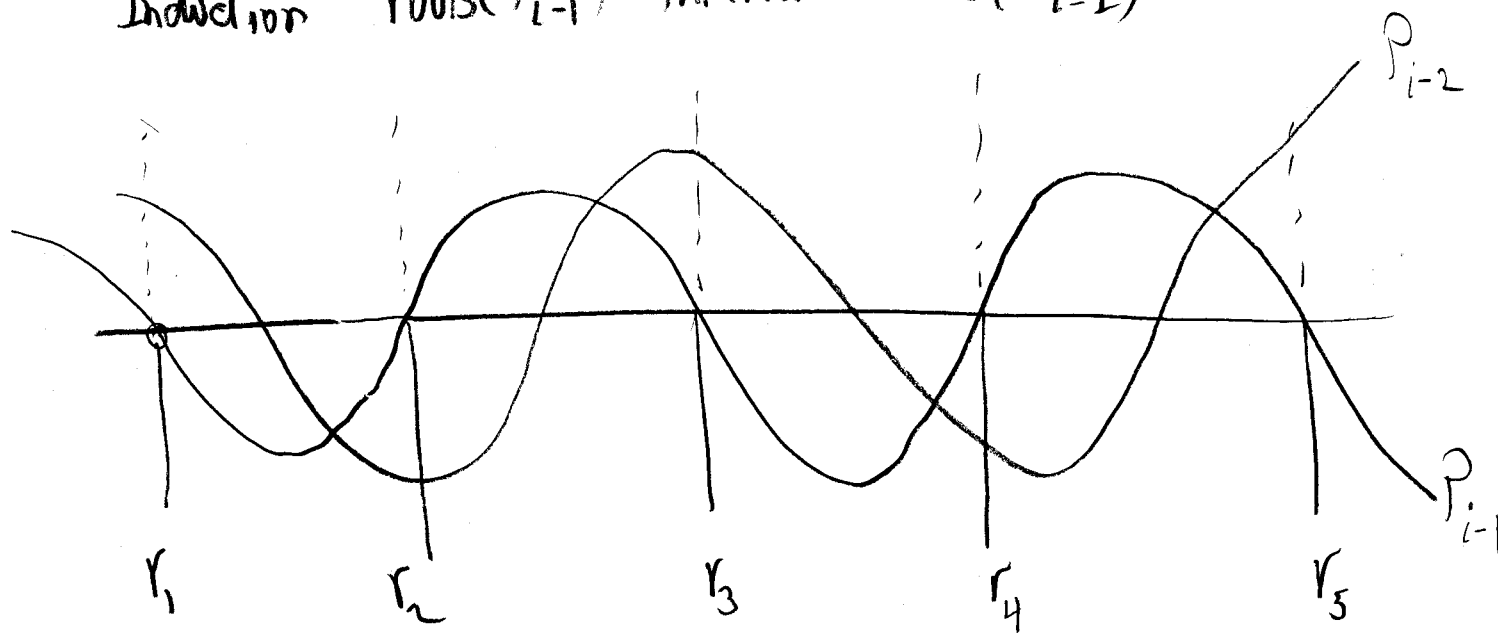
$$\Rightarrow P_{i-2}(\lambda') = 0 \Rightarrow P_0(\lambda') = 0 \text{ contra!}$$

let $r_1, \dots, r_{i-1}$ be roots of $P_{i-1}(\lambda)$

$$\text{Thus } P_i(r_k) = -b_{i-1}^2 P_{i-2}(r_k)$$

$$\text{sign}(P_i(r_k)) = -\text{sign}(P_{i-2}(r_k))$$

Induction $\text{roots}(P_{i-1})$ interlace $\text{roots}(P_{i-2})$



Between each pair r_k & r_{k+1} is a root of P_i .

We have found $i-2$ roots of P_i .

By a) & b) $\exists$ root of $P_i < r_i$
 $\exists P_i > r_{i-1}$

Thus we have found all the roots of P_i

They interlace!

Using signs to locate roots

Def Given $P_1(x), \dots, P_n(x)$ define

$$N(c) = |\{i \mid \text{Sign}(P_i(c)) \neq \text{Sign}(P_{i+1}(c))\}|$$

Thm # roots of $P_n(x) < c = N(c)$

Cost to compute $N(c)$ is $O(n)$

Divide-and-conquer Alg

$T^{n \times n}$ sym & tri diagonal

$$T = \left(\begin{array}{c|c} T_1 & \beta \\ \hline \beta & T_2 \end{array} \right) = \left(\begin{array}{c|c} \hat{T}_1 & \\ \hline & \hat{T}_2 \end{array} \right) + \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \end{array} \right)$$

Suppose we have found eigen vectors for $\hat{T}_1, \hat{T}_2$

$$\text{ie } \hat{T}_1 = Q_1 D_1 Q_1^T \quad \hat{T}_2 = Q_2 D_2 Q_2^T$$

Claim

$$T = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \left[\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} + \beta \beta^T \right] \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix}$$

Where $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

$z_1^T = \text{last row of } Q_1$

$z_2^T = \text{first row of } Q_2$

to show

$$\beta \left(\begin{array}{c|c} 1 & 1 \\ \hline 1 & 1 \end{array} \right) = \begin{pmatrix} Q_1 & \\ & Q_2 \end{pmatrix} \begin{pmatrix} z_1 z_1^T & \\ & z_2 z_2^T \end{pmatrix} \begin{pmatrix} Q_1^T & \\ & Q_2^T \end{pmatrix}$$

is "

$$= \begin{pmatrix} Q_1 & \\ & Q_2 \end{pmatrix} \left(\begin{array}{c|c} z_1 z_1^T & z_1 z_2^T \\ \hline z_2 z_1^T & z_2 z_2^T \end{array} \right) \begin{pmatrix} Q_1^T & \\ & Q_2^T \end{pmatrix}$$

consider upper-left

$$\begin{aligned} (1,1) &= Q_1 z_1 z_1^T Q_1^T = (Q_1 z_1)(Q_1 z_1)^T \\ &= \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}^T \end{aligned}$$

3-more cases.

We have reduced prob to:

Find eigens of $D + WW^T$ D is diagonal
(assume distinct eigenvalues)

Suppose $w_j = 0$ some j then e_j is an eigenvector.

$$\text{for } (D + WW^T)e_j = De_j + WW^Te_j = d_j e_j$$

note j th row & col of WW^T is zero.

$\therefore D + WW^T$ is reducible

Assume $w_j \neq 0 \forall j$

Claim $f(\lambda) = 1 + \sum_{j=1}^m \frac{w_j^2}{d_j - \lambda}$ then

$$\text{roots}(f) = \text{eigenvalues}(D + WW^T).$$

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pf Suppose eigen pair λ, g $g \neq 0$

$$(D + WW^T)g = \lambda g$$

$$(D - \lambda I)g + WW^Tg = 0$$

$$g + (D - \lambda I)^{-1}WW^Tg = 0$$

mult W^T $W^Tg + W^T(D - \lambda I)^{-1}W(W^Tg) = 0$

$$\underbrace{(1 + W^T(D - \lambda I)^{-1}W)}_{f(\lambda)}(W^Tg) = 0$$

$$f(\lambda)(W^Tg) = 0$$

Claim $f(\lambda) \neq 0$ Suppose $f(\lambda) = 0$ then

$$W^Tg = 0 \Rightarrow Dg = \lambda g$$

$$\Rightarrow g = e_j \text{ some } j$$

$$\Rightarrow w_j = 0 \text{ contra!}$$

$$f'(\lambda) = \sum_{j=1}^m \frac{d}{d\lambda} \left(w_j^2 (d_j - \lambda)^{-1} \right)$$

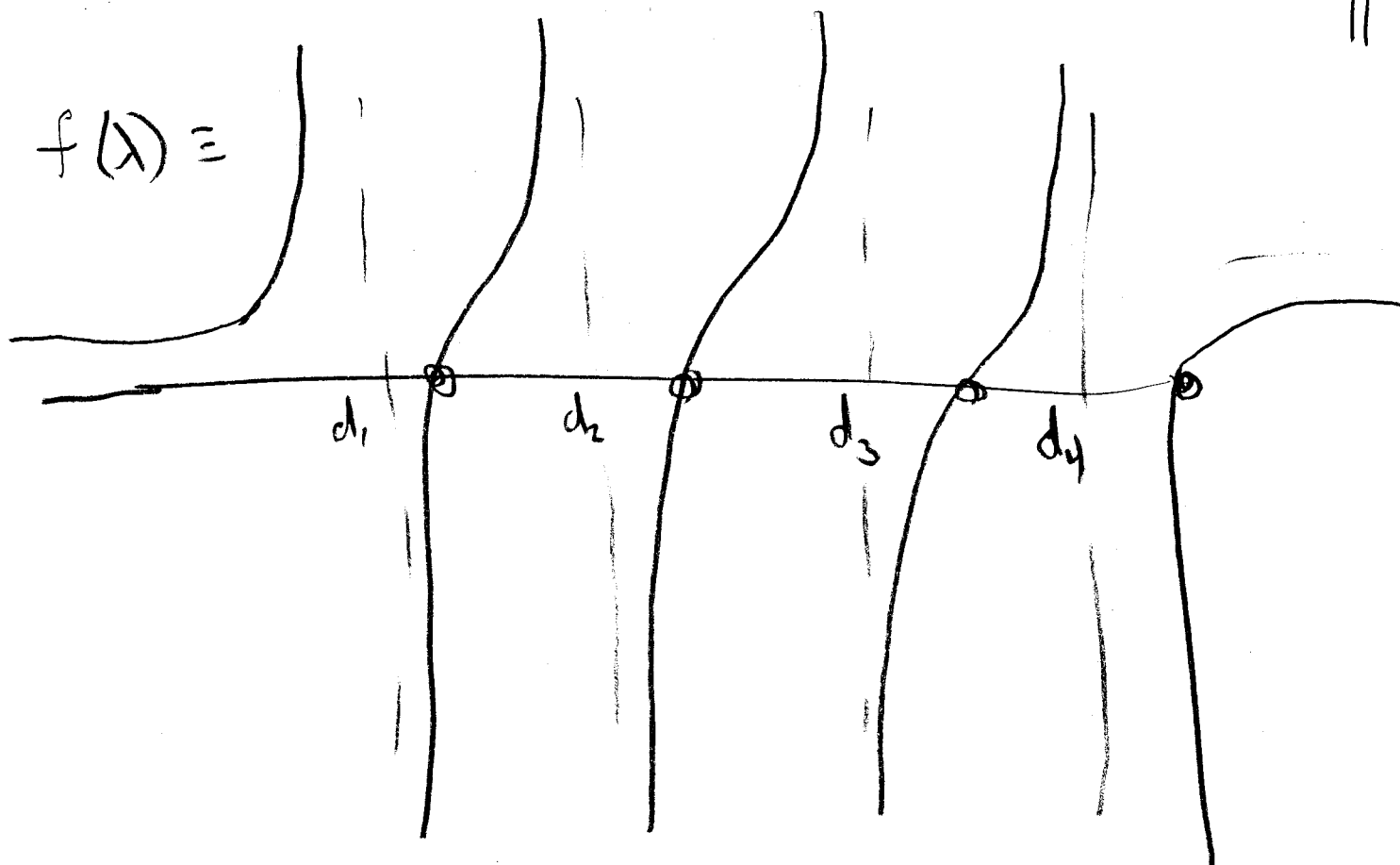
$$= \sum_{j=1}^m w_j^2 (-1) (d_j - \lambda)^{-2} (-1)$$

$$= \sum w_j^2 (d_j - \lambda)^{-2}$$

Thus $f'(\lambda) > 0$ for $\lambda \neq \text{any } d_j$

$$f(\pm\infty) = 1$$

Thus between each d_i, d_{i+1} is a root
 d_n, ∞ is a root.



Alg 1) Use Newton like scheme to find
each root in $O(1)$ iterations & $O(m)$ work/root

$O(m^2)$ total work

$$W(m) = m^2 + 2 W(m/2)$$

$$W(m) = O(m^2)$$

prob: Need (want) eigenvectors.

Thm $A = L(G)$ & $G' = G + e$ $A' = L(G')$

Eigenvalues of A & A' interlace.