15-859N

Symmetric Perron-Frobenius

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Thm (Sym Perron-Frobenius)

Let G be sym, connected, weights ≥ 0

Let $A = \mathrm{Adj}(G)$ & $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = \lambda(G)$

then

a) $\mu_1 \geq -\mu_n$

b) $\mu_1 > \mu_2$

c) $Ax = \mu_1 x$ then $\exists x$ s.t. $x > 0$

Lemma Suppose as above & $Ax = \lambda x$

if $0 \neq x \geq 0$ then $x > 0$

pf Since $A \geq 0$ & $x \geq 0$ then $Ax \geq 0$

thus $\lambda > 0$

Suppose $\exists v \in V$ s.t. $x_v = 0$.

Pick v s.t. $x_v = 0$ & $(u,v) \in E$ & $x_u \neq 0$

$$0 = \lambda x_v = (Ax)_v = \sum_{(v,u) \in E} A_{vu} x_u > 0$$

Contra!

pf (SPF) Suppose eigenvector

$$Af_i = \mu_i f_i \quad 1 \le i \le n$$

(By Spectral Thm) $\mu_1 = \max_{x \neq 0} \frac{x^T A x}{x^T x}$

Def $|x|$ s.t. $|x|_i = |x_i|$

Note $|x|^T |x| = \sum |x_i| \cdot |x_i| = \sum x_i^2 = x^T x$

$$f^T A f = \sum A_{uv} f(u) f(v)$$

$$\le \sum A_{uv} |f(u)| \cdot |f(v)|$$

$$= |f|^T A |f|$$

thus WLOG $f_1 \ge 0 \xRightarrow{\text{Lemma}} f_1 > 0$

a) Suppose $Af_n = \mu_n f_n$

$$|\mu_n| = |f_n^T A f_n| \le |f_n|^T A |f_n| \le \mu_1$$

b) Let $Af_2 = \mu_2 f_2$

By Spectral Thm pick f_2 s.t. $f_2^T f_1 = 0$

$$\mu_2 = \frac{f_2^T A f_2}{f_2^T f_2} \overset{*}{\le} \frac{|f_2|^T A |f_2|}{|f_2|^T |f_2|} \le \mu_1$$

<u>Claim</u> $*$ is strict.

<u>Subcase 1</u> $\forall v \; f_2(v) \neq 0$ ie $|f_2| > 0$

$f_2^T f_1 = 0 \Rightarrow \exists (u,v) \in E$ s.t.

$f_2(u) \cdot f_2(v) < 0$

Consider term in $f_2 A f_2$ $\quad A_{uv} f_2(u) f_2(v) < 0$

$0 < A_{uv} |f_2(u)| \cdot |f_2(v)|$

<u>Subcase 2</u> $\exists u \in V$ s.t. $f_2(u)=0$

Suppose $\mu_2 = \mu_1$

thus $Af_2 = \mu_1 f_2$ (eignvector)

$\therefore$ $|f_2|$ eigenvector with value μ_1

Lemma $\Rightarrow |f_2| > 0$ contra!