**Random Walks on Graphs**

Graph: $G = (V, E, w)$ (possibly directed)

$w_i = w(V_i) \equiv \sum_{(i,j) \in E} w_{ij}$

$p_{ij} \equiv \frac{w_{ij}}{w_i}$

Random walk on $G$:

Suppose at a given time we are at $v_i \in V$.

We move to $v_j$ with probability $p_{ij}$

E.g., $V$ = all orderings of a deck of 52 cards

$p_{ij} = \text{Prob of going from order}_i \text{ to order}_j \text{ in one shuffle}$
Question: Why do professionals play after 5 shuffles?
Important Parameters

Access time or Hitting time

\[ H_{ij} = \text{Expected time to visit } j \text{ starting at } i \]

Commute Time

\[ k(i,j) = H(i,j) + H(j,i) \]

Cover Time

Expected time to visit all nodes
max over all starting nodes

Mixing Rate (to do)
Random Walks - the Symmetric Case

Idea: Do random walk on a network of conductors?

Input: $G=(V,E,c)$ $c_{ij} = c_{ji}$

Consider a random walk
Starting at $x$ and ending at $b$.
Def $h_x$ = prob we visit a before $b$ starting at $x$, $a \neq b$.

eg $1, 2, 3, 4, 5$

\[ a \xrightarrow{} x \xrightarrow{} b \]

$h_a = 1 \& h_b = 0$

$h_x$? $h_x > \frac{1}{2}$ why?
\[ h_a = 1 \quad \& \quad h_b = 0 \]

Suppose \( X \neq a, b \)

**Claim** \( h_x = \sum_y P_{xy} h_y \)

\( P_{xy} \geq 0 \quad \& \quad \sum_y P_{xy} = 1 \)

\[ \therefore h_x \text{ is a convex comb of its neighbors!} \]

\[ h \text{ is harmonic with boundary } a, b! \]

\underline{Let's construct an identical electrical prob!}
Consider: $V_a = 1 \& V_b = 0$

$\forall x \neq a, b \quad V_x = \sum_y \frac{C_{xy}}{C_x} V_y$

but $C_{xy}/C_x = P_{xy}$! $\Rightarrow h = V$

**Thm** Set $V_a = 1 \& V_b = 0 \& x \neq a, b$

Float then $V_x =$ prob visit $a$ before $b$.

**Eg.** $1 \ 1 \ \frac{3}{4} \ \frac{1}{2} \ \frac{1}{4} \ 0$

\[ \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{array} \]

\[ a \ c \ b \]

$h_c = \frac{3}{4}$
In general may have multi 7 sinks and goals.

Thus solve in one Laplacian solve.
Interpretation of Current For Random Walk.

Consider 1 unit of potential current flow from a to b, say i. What does i_{xy} correspond to in random walk from a to b?

**Theorem:** $i_{xy} = \text{Expected net # of traversals of } Exy \text{ in random walk from a to b.}$
pf:

Def. \( U_x \equiv \text{Expected \# of visits to } x \in V \text{ before reaching } b \text{ starting at } a \)

Set \( U_b = 0 \quad x \neq a, b \)

Hw. \( U_x = \sum_{y} U_y P_{yx} \quad \text{note } \sum P_{yx} \neq 1 \)

Recall \( C_x = \sum_{y} C_{xy} \)

Note:
\[
C_x P_{xy} = C_x \left( \frac{C_{xy}}{C_x} \right) = C_{xy} = C_{yx} = C_y P_{yx}
\]

= \( C_y \left( \frac{C_{yx}}{C_y} \right) = C_y P_{yx} \)
Thus:

\[ U_x = \sum_y U_y \frac{C_y P_{yx}}{C_y} = \sum_y U_y \frac{P_{xy} C_x}{C_y} \]

\[ (\frac{U_x}{C_x}) = \sum_y P_{xy} (\frac{U_y}{C_y}) \]

Voltage: \( V_x = (\frac{U_x}{C_x}) \)

Recurrence: \( V_x = \sum_y P_{xy} V_y \)

Thus \( V_x \) is harmonic.

What is the boundary conditions?

\( V_b = 0 \) \( \quad \) \( V_a = \frac{U_a}{C_a} \) some \( U_a \).

Let \( j_{xy} = \text{current on } C_{xy} \)
\[ j_{xy} = (V_x - V_y) C_{xy} = \left( \frac{u_x}{C_x} - \frac{u_y}{C_y} \right) C_{xy} \]
\[ = u_x \left( \frac{C_{xy}}{C_x} \right) - u_y \left( \frac{C_{yx}}{C_y} \right) = u_x P_{xy} - u_y P_{yx} \]

\[ u_x P_{xy} = \text{Expected \# of } x \to y \text{ traversals} \]
\[ u_y P_{yx} = "y \to x" \]
\[ j_{xy} = \text{Expected net \# of traversals}. \]

To show: Net current flow is 1!
\[ \text{i.e. } \sum_{y} j_{xy} = 1 \]

If must be 1

This proves Thm.
Computing \( u_a \). \( \alpha = v_b \) \& \( \beta = v_n \)

Solve \( LV = \begin{pmatrix} \beta \\ -1 \end{pmatrix} \)

set \( V' = V - V_n \begin{pmatrix} \beta \\ -1 \end{pmatrix} \) i.e. set \( V_n = 0 \)

\( V_s = \begin{pmatrix} U_i \\ C_i \end{pmatrix} \) \( U_i = V_i C_i \)

we have found \( u_a \)!

\( V = E R_{ab} \)

\( u_a = C_a \cdot E R_{ab} \)
How to compute hitting time

Def \( h(x,b) \) = expected time to reach \( b \) from \( x \).

\[ h_x = h(x,b) \quad b \text{ fixed} \]

Let write a recurrence:

\[ h_b = 0 \]
\[ x \neq b \quad h_x = 1 + \sum_y h_y P_{xy} \]

Let’s think of \( h_x \) as voltage \( V_x \)

\[ V_b = 0 \quad V_x = 1 + \sum_y \frac{C_{xy}}{C_x} V_y \]
\[ C_X V_X = C_X + \sum \limits_Y C_{XY} V_Y \]

\[ C_X V_X - \sum C_{XY} V_Y = C_X \]

Graph Laplacian residual current

n-1 constraints

by adding constraint for \( V_n = b \)

\[
LV = \begin{pmatrix} C_1 \\ \vdots \\ C_{n-1} \end{pmatrix} \quad b = V_n \\
V_n = 0 \quad C = \sum C_i
\]

where \( S = C_n - C \)

Alg. for hitting time.

Solve \( LV = \begin{pmatrix} C \\ \vdots \\ C_{n-1} - C \end{pmatrix} \) return \( V_X \)
What about commute time?
set $v_i = a$ & $v_n = b$

**Solution 1**

solve $LV^b = \begin{pmatrix} c_i \\ \vdots \\ c_n-c \end{pmatrix}$ $LV^a = \begin{pmatrix} c_i-c \\ \vdots \\ c_n \end{pmatrix}$

$h(i,n) = v_i - v_n$ & $h(n,i) = v_n - v_i$

set $V = V^b - V^a$

$c(i,n) = (V^b - V^a)_i - (V^b - V^a)_n = v_i - v_n$

**Solution 2**

$L(V^b - V^a) = LV^b - LV^a = \begin{pmatrix} c_i \\ \vdots \\ c_n-c \end{pmatrix} - \begin{pmatrix} c_i-c \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} c_i \\ \vdots \\ c_n \end{pmatrix} - \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = C \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix}$
Solve $LV = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

return $C(V_1 - V_n)$ but $(V_i - V_n) = ER_{in}$

Thm $C(a, b) = C \cdot ER_{ab} = 2m \cdot ER_{ab}$

$C(a, b) = 2(n-1)$. 