Random Walks on Graphs

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Graph: $G = (V, E, w)$ (possibly directed)

$w_i = w(V_i) \equiv \sum_{(i,j) \in E} w_{ij} \quad P_{ij} \equiv w_{ij}/w_i$

Random walk on G:

Suppose at a given time we are at $V_i \in V$.

We move to V_j with probability P_{ij}

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e.g. $V \equiv$ all ordings of a deck of 52 cards

$P_{ij} \equiv$ Prob of going from order_i to order_j in one shuffle.

Question: Why do professionals play after 5 shuffles?

Important Parameters

Access time or Hitting time

H_{ij} = Expected time to visit j starting at i

Commute Time

$$K(i,j) = H(i,j) + H(j,i)$$

Cover Time

Expected time to visit all nodes

max over all starting nodes

Mixing Rate (to do)

Random Walks - the Symmetric Case

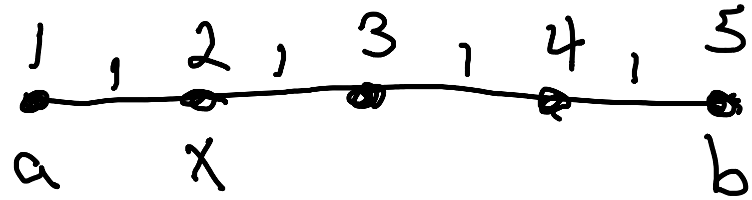
Idea: Do random walk on a network of conductors!

Input: $G=(V,E,c)$ $c_{ij}=c_{ji}$

Consider a random walk starting at x and ending at b.

<u>Def</u> $h_x \equiv$ prob we visit a before b starting at x, $a \neq b$.

eg



$h_a = 1$ & $h_b = 0$

h_2? $h_2 > 1/2$ why?

$h_a = 1$ & $h_b = 0$

Suppose $x \neq a, b$

<u>Claim</u> $h_x = \sum_y P_{xy} h_y$

$P_{xy} \geq 0$ & $\sum_y P_{xy} = 1$

$\therefore$ h_x is a convex comb of its neighbors!

h is harmonic with bdary a,b!

Lets construct an identical electrical prob!

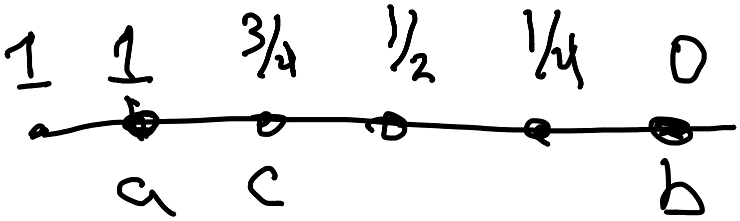
Consider: $V_a = 1$ & $V_b = 0$

$\forall x \neq a, b \quad V_x = \sum_y \frac{C_{xy}}{C_x} V_y$

but $C_{xy}/C_x = P_{xy}$! $\Rightarrow h = V$

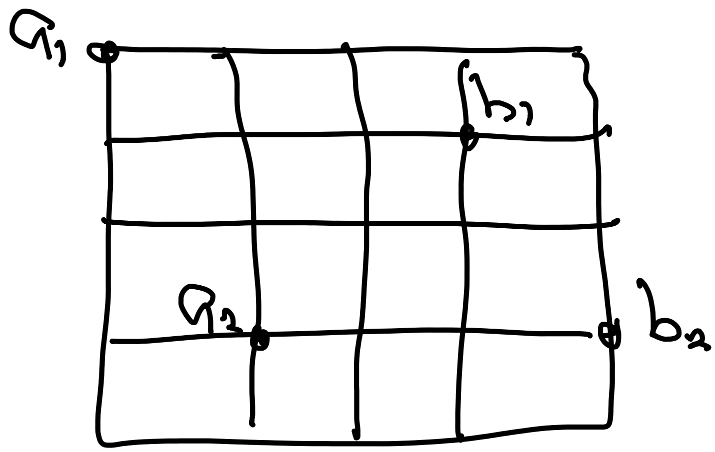
<u>Thm</u> Set $V_a = 1$ & $V_b = 0$ & $x \neq a, b$ float then V_x = prob visit a before b.

e.g.



$h_c = 3/4$

In general may have multi sinks and goals.



Thus solve in one Laplacian solve.

Interpretation of Current for Random Walk.

Consider 1 unit of potential current flow from a to b, say i

what does i_{xy} correspond to in random walk from a to b?

<u>Thm</u> i_{xy} = Expected net # of traversals of E_{xy} in random walk from a to b.

pf:

Def: $u_x \equiv$ Expected # of visits to $x \in V$ before reaching b starting at a

Set $u_b = 0 \quad x \neq a, b$

HW: $u_x = \sum_y u_y P_{yx}$ note $\sum P_{yx} \neq 1$

Recall $C_x = \sum_y C_{xy}$

note:

$$C_x P_{xy} = C_x \left(\frac{C_{xy}}{C_x} \right) = C_{xy} = C_{yx}$$

$$= C_y \left(\frac{C_{yx}}{C_y} \right) = C_y P_{yx}$$

thus:

$$u_x = \sum_y u_y \frac{c_y P_{yx}}{c_y} = \sum_y u_y \frac{P_{xy} c_x}{c_y}$$

$$\therefore \left(\frac{u_x}{c_x}\right) = \sum_y P_{xy} \left(\frac{u_y}{c_y}\right)$$

Voltage: $V_x = \left(\frac{u_x}{c_x}\right)$

recurance $V_x = \sum_y P_{xy} V_y$

thus V_x is harmonic!

what is the bdary conditions?

$V_b = 0$; $V_a = \frac{u_a}{c_a}$ some u_a.

Let $j_{xy} \equiv$ current on c_{xy}

$$j_{xy} = (V_x - V_y)C_{xy} = \left(\frac{u_x}{C_x} - \frac{u_y}{C_y}\right)C_{xy}$$

$$= u_x\left(\frac{C_{xy}}{C_x}\right) - u_y\left(\frac{C_{yx}}{C_y}\right) = u_x P_{xy} - u_y P_{yx}$$

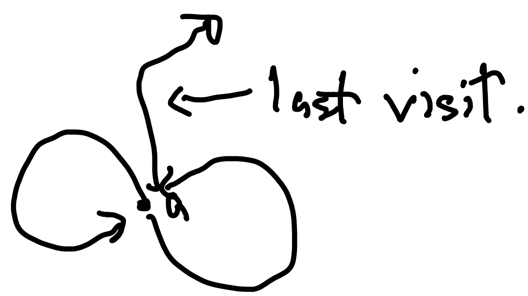
$u_x P_{xy}$ = Expect # of x to y traversals

$u_y P_{yx}$ = " "y to x" ".

j_{xy} = Expected net # of traversals.

To show: Net current flow is 1!

i.e. $\sum_y j_{ay} = 1$



It must be 1

This proves Thm.

Computing u_a. $a = v_1$ & $b = v_n$

Solve $LV = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ -1 \end{pmatrix}$

set $V' = V - V_n \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ i.e. set $V_n = 0$

$V_1 = \left(\frac{U_1}{C_1} \right)$ $U_1 = V_1 C_1$

we have found u_a!

$V = ER_{ab}$

$u_a = C_a \cdot ER_{ab}$

How to compute hitting time

<u>Def</u> $h(x,b) \equiv$ expected time to reach b from x.

$h_x = h(x,b)$ b fixed

Let write a recurrence:

$$h_b = 0$$

$$x \neq b \quad h_x = 1 + \sum_y h_y P_{xy}$$

Let's think of h_x as voltage V_x

$$V_b = 0 \quad V_x = 1 + \sum_y \frac{C_{xy}}{C_x} V_y$$

$$C_x V_x = C_x + \sum_y C_{xy} V_y$$

$$\underbrace{C_x V_x - \sum C_{xy} V_y}_{\text{Graph Laplacian}} = \underbrace{C_x}_{\text{residual current}}$$

$n-1$ constraints

by adding constraint for $V_n = b$

$$\underset{V_n=0}{LV} = \begin{pmatrix} c_1 \\ \vdots \\ c_{n-1} \\ \delta \end{pmatrix} \qquad \begin{array}{l} b = V_n \\ c = \sum c_i \end{array}$$

where $\delta = c_n - c$

<u>Alg</u> for hitting time.

solve $\underset{V_n=0}{LV} = \begin{pmatrix} c_1 \\ \vdots \\ c_n - c \end{pmatrix}$ return V_x

What about commute time?

set $v_1 = a$ & $v_n = b$

<u>Solution 1</u>

solve $Lv^b = \begin{pmatrix} c_1 \\ \vdots \\ c_n - c \end{pmatrix}$ $Lv^a = \begin{pmatrix} c_1 - c \\ \vdots \\ c_n \end{pmatrix}$

$h(1,n) = v_1^b - v_n^b$ & $h(n,1) = v_n^a - v_1^a$

set $v = v^b - v^a$

$C(1,n) = (v^b - v^a)_1 - (v^b - v^a)_n = v_1 - v_n$

<u>Solution 2</u>

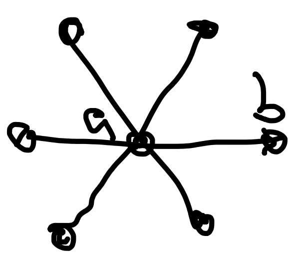
$L(v^b - v^a) = Lv^b - Lv^a =$

$$\begin{pmatrix} c_1 \\ \vdots \\ c_n - c \end{pmatrix} - \begin{pmatrix} c_1 - c \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} c \\ 0 \\ \vdots \\ 0 \\ -c \end{pmatrix} = c \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}$$

Solve $LV = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}$

return $C(V_1 - V_n)$ but $(V_1 - V_n) = ER_{1n}$

<u>Thm</u> $C(a,b) = C \cdot ER_{ab} = 2m\, ER_{ab}$



$C(a,b) = 2(n-1)$.