15-859N

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Computing Min Energy Flow using Cycle updates.

Goal Solve $Lv = d$ $L\bar{v} = d$

where $L = B^T C B$ $R = C^{-1}$

<u>Recall</u> Potential flow $f_p = CBv$

Circulations $B^T f = 0$

<u>Thompson</u> Min energy flow is a potential flow

<u>pf</u> fact $f_p^T R f_c = 0$

We think in terms of Primal & Dual

Primal $\{ f \mid B^T f = d \}$

<u>Dual</u> Potentials

Dual

Object $\equiv 2V^Td - V^TLV \equiv E(V)$

<u>Lemma</u> $E(\bar{V}+V) = \bar{V}^Td - V^TLV$

thus $E(V)$ is a maximum at $V=\bar{V}$

Consider $2V^Td - V^TLV = E(V)$.

$$\nabla E = 2d - 2LV$$

$$\nabla E(V) = 0 \Rightarrow LV = d$$

min or max at $\bar{V}$ st $L\bar{V} = d$

Substitue $\bar{V}$

$$E(\bar{V}) = 2\bar{V}^Td - \bar{V}^TL\bar{V} = 2\bar{V}^Td - \bar{V}^Td = \bar{V}^Td$$

but $\bar{V}^Td = \bar{V}^TL\bar{V} = \bar{V}^T\partial_1 \underline{C\,\partial_1^T\bar{V}_1} = \bar{V}^T\partial_1 f_p$

$$= (RC\,\partial_1^T\bar{V})^T f_p$$
$$= f_p^T R f_p$$

Claim $\bar{v}^T d$ is a max

$$2(\bar{v}+v)^T d - (\bar{v}+v)^T L(\bar{v}+v)$$

$$= 2\bar{v}^T d + 2v^T d - \bar{v}^T L\bar{v} - \underbrace{2v^T L\bar{v}}_{2v^T d} - v^T L v$$

$$= \bar{v}^T d + 0 - v^T L v$$

$$= \bar{v}^T d - v^T L v \quad \text{where } v^T L v \geq 0$$

$$E(v) \leq E(\bar{v}) = E(f_p) \leq E(f_p + f_c)$$

Idea: Find some f st $B^T f = d$

Keep adding circulation to min energy.

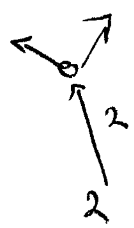
<u>Pick</u>: ST T of G

In particular: T LSST

<u>Claim</u> $\exists!$ flow f st $\partial_1 f = d$

$f(e) = 0$ for $e \notin T$. say f_0

Do a BFS from a leaf of T.



Tree induced voltage

$$V_T(f) = V(a) = \sum_{e \in P(s,a)} f(e) r_e \quad \text{some fixed } s \in V.$$

<u>Def</u> Tree Condition Number

$T \subseteq G$

$e \in G \setminus T$ we defined $ER_T(e) = \sum_{e \in R(a,b)} r_e$

KOSZ: $R_e = ER_T(e) + r_e$

Tree Cond. #: $\tau(T) = \sum_{e \in E \setminus T} \frac{R_e}{r_e} =$

$$STr_T(e) = \frac{ER_T(e)}{r_e}$$

$$STr_T(G) = \sum_{e \in G} STr_T(e)$$

$$\tau(T) = \underbrace{\sum_{e \in E \setminus T} str_T(e)}_{STr_T(G) - (n-1)} + m - n + 1$$

$$STr_T(G) - (n-1) + m - (n+1)$$

$$STr(G) + m - 2n + 2$$

Let f_e be unit flow on cycle C_e

Goal: $\min_{\alpha} (f+\alpha f_e)^T R (f+\alpha f_e)$

Claim same as proj onto space

$\{f : f^T R f_e = 0\}$ R-orthogonal

ie α st $(f+\alpha f_e)^T R f_e = 0$

$$f^T R f_e + \alpha f_e^T R f_e = 0$$

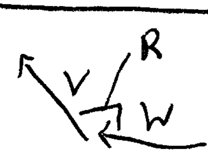
$$\alpha = -\frac{f^T R f_e}{R_e}$$

Def $\Pi_e(f) = f + \alpha f_e$

$$E(f) - E(\Pi_e(f)) = \alpha^2 f_e^T R f_e = \frac{(f^T R f_e)^2}{R_e}$$

R-orth

Note



$$(V+W)^T R (V+W) - W^T R W = V^T R V$$

KOSZ - Min Energy Flow

Iput: $L_G X = d$ $\quad K$ integer

1) $T \equiv LSST(G)$
2) Comput tree flow f_0 s.t. $B^T f = d$
   Set $f = f_0$
3) For K rounds
   a) $e \in E \setminus T$ proportional $\frac{R_e}{r_e}$
   b) set $f = \Pi_e(f)$
4) Return f

Tree Condition number $\quad e = (a, b)$

<u>Def</u> $e \in E \setminus T$ $\quad ER_T(e) = \sum_{e' \in aPb} r_e$

KOSZ: $R_e = ER_T(e) + r_e$

Condition Number of T

$$\gamma(T) = \sum_{e \in E \setminus T} \frac{R_e}{r_e}$$

Energy decrease per round

flow f, $e \in E \setminus T$, unit flow on $C_e = f_e$

<u>Def</u> $\pi_e(f) = f + \alpha f_e$ (new flow)

<u>Lemma</u> $E(f) - E(\pi_e(f)) = \dfrac{(f^T R f_e)^2}{R_e}$

<u>Lemma</u> (Tree Gap) $B^T f = d$ & $v = Volt_T(f)$

$$gap(f, v) = \sum_{e \in E \setminus T} \frac{(f^T R f_e)^2}{r_e}$$

<u>Lemma</u> (Tree Gap) $B^T f = d$ & $v = v_T(f)$

$$gap(f,v) = \sum_{e\in E\setminus T} \frac{(f^T R f_e)^2}{r_e}$$

$$gap(f,v) = f^T R f - (2v^T d - v^T L v)$$

(calculation)

$$= (Rf - \partial_1^T v) R^{-1} (Rf - \partial_1^T v)$$

$$= \sum_{e\in E} \frac{1}{r_e}\left(f(e) r_e - \Delta_v(e)\right)^2$$

$e \in T$ then $\Delta_v(e) = f(e) r_e$

$e = (a,b)$

$$\Delta_v(e) = \sum_{e\in P(a,b)} f(e)\, r_e$$

$$f(e)\, r_e - \sum_{e'\in P} f(e')\, r_{e'} = f^T R f_e$$

Lemma (Expected Progress)

$$E_X\left[E(f_i) - E(f_{i-1}) \,\middle|\, gap(f_{i-1}, V_{i-1}) \right] = *$$

$$= -\frac{gap(f_{i-1}, V_{i-1})}{\tau(T)}$$

Pf

$$* = E_X\left[\sum_{e \in E \setminus T} \overbrace{p_e \left(-\frac{(f_{i-1}^T R f_e)^2}{R_e} \right)}^{**} \,\middle|\, gap(f_{i-1}, V_{i-1}) \right]$$

$$p_e = \frac{1}{\tau} \cdot \frac{R_e}{r_e}$$

$$** \quad \sum_{e \in E \setminus T} -\frac{1}{\tau} \frac{\cancel{R_e}}{r_e} \left(\frac{f_{i-1}^T R f_e}{\cancel{R_e}} \right)^2$$

$$\frac{1}{\tau} \sum -\frac{(f_{i-1}^T R f_e)^2}{r_e} = -\frac{1}{\tau} gap(f_{i-1}, V_{i-1})$$

Bounding Initial Energy

HW: $L_G \preceq \mathrm{str}_T(G)\, L_T$

$$\mathcal{E}(f_0) = 2d^T p - p^T L_T p \qquad L_T p = d$$

$$\leq 2d^T p - \left(\frac{1}{\mathrm{str}}\right) p^T L_G p$$

$$\leq (\mathrm{str}) \left[2d^T (p/\mathrm{str}) - (p/\mathrm{str})^T L_G (p/\mathrm{str}) \right]$$

$$\leq \mathrm{str}_T(G)\, \mathcal{E}(f^*)$$

$$\mathbb{E}\left[E(f_i) - E(f_*)\right] \leq \left(1 - \frac{1}{\tau}\right)\mathbb{E}\left[E(f_{i-1}) - E(f_*)\right]$$

<u>Lemma</u> $\mathbb{E}\left[E(f_k) - E(f_*)\right] \leq \left(1 - \frac{1}{\tau}\right)^k E(f_0)$

$$\leq \left(1 - \frac{1}{\tau}\right)^k st_r(G) \cdot E(f_*)$$