Matrix Concentration Bounds

Chernoff Bound:

<u>Thm</u> Let $X_1, \dots, X_n$ be independent random real vars with $0 \le X_i \le R$

$\mu_{min} \le \sum E[X_i] \le \mu_{max}$ then

$\forall \delta \ge 0$

$$+) \quad Pr\left[\sum X_i \ge (1+\varepsilon)\mu_{max} \right] \le \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}} \right)^{\mu_{max}/R}$$

$$\le \begin{cases} e^{-\delta^2 \mu_{max}/3R} & \delta \le 1 \\ e^{-\delta \mu_{max}/3R} & \delta > 1 \end{cases}$$

$$-) \quad Pr\left[\sum X_i \le (1-\varepsilon)\mu_{min} \right] \quad \left(\frac{e^{-\delta}}{(1+\delta)^{1+\delta}} \right)^{\mu_{min}/R}$$

$$\le e^{-\delta^2 \mu_{min}/2R} \quad \delta \le 1$$

Thm (Tropp) Let $X_1, \dots, X_k$ in ind rand sym $d \times d$ matrices & $0 \preceq X_i \preceq R$

If $\mu_{min} \cdot I \preceq \sum E(X_i) \preceq \mu_{max} \cdot I$

then $0 \leq \delta \leq 1$

+) $$Pr\left[\lambda_{max}\left(\sum X_i\right) \geq (1+\delta)\mu_{max}\right]$$

$$\leq d \cdot \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu_{max}/R} \leq d \begin{cases} e^{-\delta^2 \mu_{max}/3R} \\ e^{-\delta \mu_{max}/3R} \end{cases}$$

for $\delta \leq 1$ & $\delta > 1$

-) $$Pr\left[\lambda_{min}\left(\sum X_i\right) \geq (1-\delta)\mu_{min}\right]$$

$$\leq d \cdot \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{\mu_{min}/R} \leq d \; e^{-\delta^2 \mu_{min}/2R}$$

for $\delta \leq 1$

Tropp $\Rightarrow$ (AW)

In AW hypthesis

1) Y_i are iid (OK)

2) $E(Y_i) = Z$

3) $Y_i \preceq \mu Z$

for 2) Set $\bar{Y}_i = Z^{-1/2} Y_i Z^{-1/2}$

$$E(\bar{Y}_i) = \sum p(i)$$

$$E(\bar{Y}_i) = \sum P_r[Y_i = Y] Z^{-1/2} Y Z^{-1/2}$$

$$= Z^{-1/2} \left(\sum P_r[Y_i = Y] Y \right) Z^{-1/2}$$

$$= Z^{-1/2} Z Z^{-1/2} = I$$

3) $\bar{Y}_i = Z_i^{-1/2} Y_i Z_i^{-1/2} \preceq Z_i^{-1/2} \mu Z Z^{-1/2}$

$\preceq \mu I$

Lets do some examples:

1) $$E_i = \begin{array}{c} \begin{pmatrix} 0 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 0 \end{pmatrix} \\ \underbrace{\qquad\quad}_{i} \qquad\qquad \end{array}$$

Y = pick random $i \in [1, \dots, d]$
return $d E_i$

$$E(Y) = \sum_{i=1}^{d} p(i)\, d\, E_i = \sum E_i = I$$

$$\sum_{i=1}^{k} E(Y_i) = k\, I$$

$$k I \preceq \sum_{i=1}^{n} E(Y_i) \preceq k I$$

$$\Pr\left[\lambda_{max}\left(\sum Y_i\right) \geq (1+2)k\right]$$

$$\leq d e^{-2k/3d} \qquad (*)$$

Set $k = cd \log d$

$$(*) = d e^{-2cd\log d/3d}$$

$$= d \cdot d^{-2/3 c} = d^{1-\frac{2}{3}c}$$

$c = 3$ $\quad = d^{-1}$

Ex 2)

$$L = L(G) \quad H_{ij} = \begin{array}{cc|l} i & j & \\ \hline \end{array} \begin{array}{c} \\ i \\ j \end{array}$$

$$L = L(G) \quad H_{ij} = \begin{array}{c} \begin{array}{cc} i & j \end{array} \\ \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{array} \begin{array}{c} i \\ j \end{array} = H_e$$

$$L = \sum_{e \in E} H_e \quad \text{(unweighted case)}$$

$Y \equiv$ pick random $e \in E$ return $m H_e$

$$E(Y) = \sum_{i=1}^{m} P(i)\, m\, H_e = \sum H_e = L.$$

(Simple CB) $X_1, \ldots X_n$ are iid 0-1 RVS

with $E(X_i)=p$ & $X = \Sigma X_i$ $E(X) = \mu = p \cdot n$

$P_r[X \geq (1+\beta)] < e^{-\beta\mu/3}$ $\beta \geq 1$

pf $P_r[X \geq m] = P_r[e^{Xt} \geq e^{mt}]$

$\leq e^{-tm} E[e^{tX}]$ Markov

$= e^{-tm} E[e^{t(X_1 + \cdots X_n)}]$ (for matrix Golden-Thom)

$= e^{-tm} E[\prod e^{tX_i}]$

$= e^{-tm} \prod E[e^{tX_i}]$ indep

$= e^{-tm} \prod_n (e^t p + 1 - p)$

$= e^{-tm} (e^t p + 1 - p)^n$

$$\min_t \left[e^{-tm} (e^t p + 1 - p)^n \right] = \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}} \right)^{pn}$$

for $m = (1+\delta) p \cdot n$