Golden-Thompson Ineq 15-859N

9/26/16

<u>Thm</u> Sym A & B then

$$Tr(e^{A+B}) \leq Tr(e^A e^B)$$

<u>note</u> $e^{A+B} = e^A e^B$ iff $AB = BA$

In general: $Tr(e^{A+B+C}) \not\leq Tr(e^A e^B e^C)$

<u>Lie Product Forumla</u>

$$e^{A+B} = \lim_{N \to \infty} \left(e^{A/N} e^{B/N}\right)^N$$

<u>Pf</u> Compare $X_N = e^{(A+B)/N}$

as $N \to \infty$ $\quad Y_N = e^{A/N} e^{B/N}$

Taylor Exp

$$X_N = 1 + \frac{A+B}{N} + O(N^{-2})$$

$$Y_N = \left[1 + \frac{A}{N} + O(N^{-2})\right]\left[1 + \frac{B}{N} + O(N^{-2})\right]$$

$$= \left[1 + \frac{A}{N} + \frac{B}{N} + O(N^{-2})\right]$$

Thus $X_N - Y_N = O(N^{-2})$

Lets compare $X_N^N - Y_N^N$?

<u>Def</u> $\| \ \|$ is a matrix norm if

1) $\|A\| \geq 0$ 2) $\|A+B\| \leq \|A\| + \|B\|$

3) $\|A \cdot B\| \leq \|A\| \|B\|$

Claim $\|X^N - Y^N\| \leq N M^{N-1} \|X - Y\|$

where $M = \max(\|X\|, \|Y\|)$ & $\|\cdot\|$ norm

$$X^N - Y^N = (X^N - X^{N-1}Y) + (X^{N-1}Y - X^{N-2}Y^2)$$

$$+ \cdots (XY^{N-1} - Y^N)$$

$$= X^{N-1}(X-Y) + X^{N-2}(X-Y)Y$$

$$+ \cdots (X-Y)Y^{N-1}$$

$$\leq N M^{N-1} \|X - Y\|$$

To Show

$$\lim_{N\to\infty} (X_N^N - Y_N^N) = 0$$

Note $\|e^A\| = \|I + A + \frac{A^2}{2!} + \cdots\|$

$$\le \|I\| + \|A\| + \frac{\|A^2\|}{2} + \cdots$$

$$\le \|I\| + \|A\| + \frac{\|A\|^2}{2} + \cdots$$

$$= e^{\|A\|}$$

Thus

$$X_N \le e^{\|A+B\|/N} \le e^{(\|A\|+\|B\|)/N}$$

$$Y_N \le \|e^{A/N} \cdot e^{B/N}\| \le e^{(\|A\|+\|B\|)/N}$$

$$M^N = \max(\|X\|, \|Y\|)^N \le e^{\|A\|+\|B\|}$$

$$\|X_N^N - Y_N^N\| \le N e^{\|A\|+\|B\|} O(N^{-2})$$

$$= O(1/N)$$

Def $|v| = (v^T v)^{1/2}$

<u>Fact</u> $|X| = |X \; X|$

<u>Prop 3</u> X may not be sym $m \in \mathbb{Z}$

$$|\operatorname{tr}(X^m)| \leq \operatorname{tr}(|X|^m)$$

(Weyl's Majorant Thm) $A^{n \times n}$

singular values $s_1 \geq \cdots \geq s_n$

eigenvalues $|\lambda_1| \geq \cdots \geq |\lambda_n|$

$f : \mathbb{R}^+ \to \mathbb{R}^+$ s.t. $f(e^t)$ is convex & incr

then

$$\sum_{i=1}^{k} f|\lambda_i| \leq \sum_{i=1}^{k} f(s_i) \qquad 1 \leq k \leq n$$

we only need:

$$\sum_{i=1}^{n} |\lambda_i|^M \leq \sum_{i=1}^{n} s_i^M$$

(Singular Value Decomp) for $A^{n\times n}$

$\exists$ unitary U, V st $A = U \Lambda V^T$

where $\Lambda = \begin{pmatrix} s_1 & & \\ & \ddots & \\ & & s_n \end{pmatrix}$ $s_i \geq 0$.

<u>no pf</u>

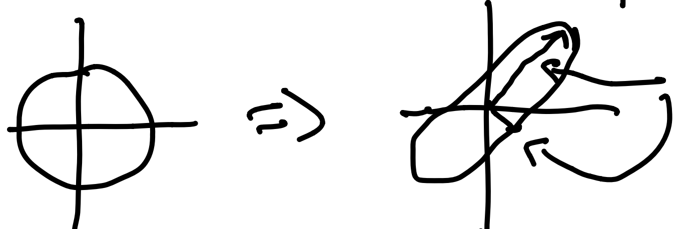
$$A = U \Lambda V^T V U^T U V^T$$

$$= U \Lambda U^T (U V^T) = M \cdot P$$

M spsd & P rigid transformation

(<u>Polar Decomp</u>) $\exists M, P$ st $A = MP$

eg $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \Rightarrow \lambda_1 = \lambda_2 = 1$ | $s_1 \cdot s_2 = 1$

A = 

Weyl: $\Rightarrow$ $\lambda_1 \leq s_1$ &

$$\lambda_1 + \lambda_2 \leq s_1 + s_2$$

<u>Note</u> $AA^T = MPP^T M = M^2 = \mathcal{U}\Lambda^2\mathcal{U}^T$

$\therefore (AA^T)^{1/2} = \mathcal{U}\Lambda\mathcal{U}^T$

Thus $\lambda(|A|) = \mathrm{sing}(A)$

Weyl $\Rightarrow$ Prop 3

$$|\mathrm{tr}(X^m)| = |\sum \lambda_i^m| \leq \sum |\lambda_i|^m$$

$$\leq \sum s_i^m = \mathrm{tr}(|X|^m)$$

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pf GT

Let $X = e^{A/2^N}$ $Y = e^{B/2^N}$

<u>to show</u> $\mathrm{tr}((XY)^{2^N}) \leq \mathrm{tr}(X^{2^N} \cdot Y^{2^N})$

(unmix the X & Y)

$$\mathrm{tr}(e^{A+B}) = \mathrm{tr}((XY)^{2^N}) \quad N \to \infty$$

$$\mathrm{tr}(X^{2^N} \cdot Y^{2^N}) = \mathrm{tr}(e^A e^B)$$

$$\operatorname{tr}\left((XY)^{2^N}\right) \leq |\operatorname{tr}[(XY)^{2^N}]|$$

$$\leq \operatorname{tr}\left[|XY|^{2^N}\right]$$

$$= \operatorname{Tr}\left[(YXXY)^{2^{N-1}}\right]$$

$$\operatorname{Tr}\left[YX^2YYX^2YY\cdots X^2Y\right]$$

$$= \operatorname{tr}\left[X^2Y^2\cdots X^2Y^2\right] \qquad \left[\operatorname{tr}(UV) = \operatorname{Tr}(VU)\right]$$

$$= \operatorname{Tr}\left((X^2Y^2)^{2^{N-1}}\right)$$

$$\leq \operatorname{tr}\left((X^4Y^4)^{2^{N-2}}\right) \qquad (\text{repeating})$$

$$\overset{?}{\leq} \operatorname{tr}\left(X^{2^N} Y^{2^N}\right)$$

Done!!