

Fiedler's Thm on Nodal Domains

G connected weighted graph Laplacian L_G

Eigenvalues $0 = \lambda_1 < \lambda_2 < \dots < \lambda_n$

vectors $v_1, \dots, v_n$

Note $v_i: V \rightarrow \mathbb{R}$

let $f: V \rightarrow \mathbb{R}$

Def (Nodal Domains of f)

Let $f^+ = \{v \in V \mid f(v) > 0\}$

$f^- = \{v \in V \mid f(v) < 0\}$

Induced graph on $W \subseteq V$ is

$$G(W) = (W, E_W) \quad E_W = \{(v, u) \in E \mid v, u \in W\}$$

Nodal Domain is a connected component of either $G(f^+)$ or $G(f^-)$

Def (Weak Nodal Domains of f)

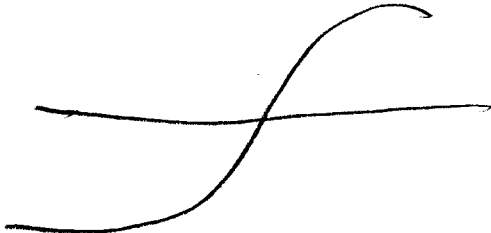
$$W_f^+ = \{v \in V \mid f(v) \geq 0\}$$

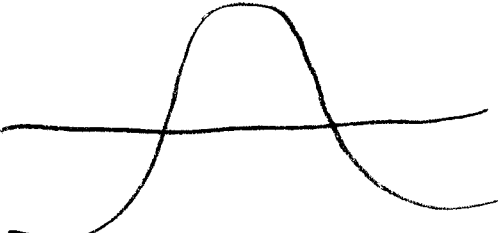
WND $\equiv$ connected components of W_f^+ .

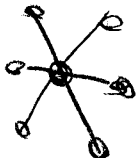
We could also use $W_f^- = \{v \in V \mid f(v) \leq 0\}$

EG. $G = P_n$

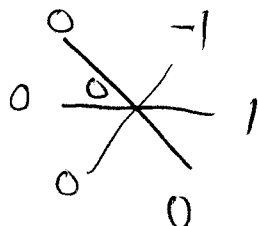
$V_1 \equiv (1, \dots, 1)$ one ND.

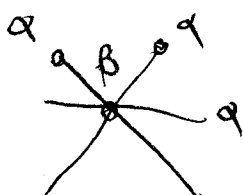
V_2  $\Rightarrow$ 2-NDs.

V_3  3-NDs.

$G = S_n$  $n-1$ leaves

$$\lambda(S_n) = (0, 1, \dots, 1, \lambda_n)$$



$$\lambda_n = \text{  } \quad (n-1)\alpha = -\beta$$

for $\lambda_1, \dots, \lambda_{n-1}$ one WND

but 2 to $n-1$ ND.

Thm (Fiedler 75) If G is Weighted connected

& $\lambda(L_G) \equiv 0 = \lambda_1 < \dots \leq \lambda_n$ with

Vectors $v_1, \dots, v_n$ then

for $k \geq 2$ V_k has $\leq k-1$ weak nodal domains

Preliminaries

4

Perron-Frobenius for generalized Laplacian

Def M is generalized Laplacian if

- 1) $M_{ij} \leq 0$ for $i \neq j$
- 2) The ^{nonzero} off diagonals form a connected ^{undirected} graph
- 3) $M_{ii} ?$

Cor PF If M is GL and λ_1 smallest eigenvalue with vector v_1 , then λ_1 has multiplicity one and v_1 strictly positive or negative.

pf pick τ s.t. $A = \tau I - M \geq 0$

Let α_1 max eigenvalue of A with strictly positive v_1 by PF Thm.

then $\lambda_1 = \nabla - \alpha_1$, with eigenvector v_1 .

Thm (Eigenvalue Interlacing)

A sym $n \times n$ matrix & B is A with same row & column deleted then

$$\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \dots \leq \alpha_{n-1} \leq \beta_{n-1} \leq \alpha_n$$

where $\alpha_1 \leq \dots \leq \alpha_n$ & $\beta_1 \leq \dots \leq \beta_{n-1}$ eigens of A & B .

Pf (Courant-Fischer)

Hint Let $\tilde{B}$ be B with zero column & row added.

Cor $\alpha_i \leq \beta_i$

Cor If B is a principle submatrix of A

then $\alpha_i \leq \beta_i$

(proof of Fiedler 75)

6

Note $W_k \neq \emptyset$ $V_k \neq 0$ & $V_k \perp \mathbb{1}$.

Suppose $G(W_k)$ has t connected components.

We reorder vertices by component they belong.

$$L_G = \begin{pmatrix} B_1 & 0 & 0 & C_1 \\ 0 & B_2 & 0 & \\ \vdots & \vdots & 0 & \\ 0 & 0 & B_t & C_t \\ C_1^T & & C_t^T & D \end{pmatrix} \quad \begin{pmatrix} x_1 \\ \vdots \\ x_t \\ y \end{pmatrix}$$

x_i, B_i are vertices in i th ND.

Vertices in $D = \{v \in V \mid v_k(v) < 0\}$ = vertices of Y .

$$x_i \geq 0 \text{ \& \; } y < 0$$

$$\begin{pmatrix} B_1 & & C_1 \\ & \ddots & \\ & & B_t & C_t \\ C_1^T & & C_t^T & D \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_t \\ y \end{pmatrix} = \lambda_K \begin{pmatrix} x_1 \\ \vdots \\ x_t \\ y \end{pmatrix}$$

note $B_i x_i + C_i y = \lambda_K x_i$

where $C_i \leq 0 \wedge y < 0 \Rightarrow C_i y \geq 0 \text{ \& } C_i y \neq 0$

$x_i \geq 0 \text{ \& } x_i \neq 0 \text{ \& } B_i x_i \neq \lambda_K x_i$

$B_i x_i \leq \lambda_K x_i \Rightarrow x_i^T B_i x_i \leq \lambda_K x_i^T x_i$

Thus $\lambda_1(B_i) \leq \lambda_K$

$\lambda_1(B_i) = \arg \min_{x \neq 0} \frac{x^T B_i x}{x^T x} = V_1(B_i)$

Case 1 x_i is not strictly positive $\Rightarrow$

$\Rightarrow x_i$ is not eigenvector of smallest value

$\Rightarrow \lambda_1(B) < \lambda_K$

Case 2 x_i strictly positive

then $x_i^T C_i y > 0$

$$x_i^T B_i x_i < x_i^T B_i x_i + x_i^T C_i y = \lambda_k x_i^T x_i$$

Thus $\lambda_1(B_i) < \lambda_k$ ie $B_i w_i = \mu_i w_i$
 $\mu_i < \lambda_k$

Consider Principle Submatrix

$$\left[\begin{array}{c|c} B_1 & 0 \\ \hline 0 & B_t \end{array} \right] \begin{pmatrix} 0 \\ \vdots \\ \mu_i \\ \vdots \\ 0 \end{pmatrix} = \mu_i \begin{pmatrix} 0 \\ \vdots \\ \mu_i \\ \vdots \\ 0 \end{pmatrix}$$

We have t eigenvalues $\mu_1, \dots, \mu_t < \lambda_k$

WLOG $\mu_1 \leq \dots \leq \mu_t$

$$\lambda_t \leq \mu_t \Rightarrow t \leq k-1.$$

9

$$\text{Let } f: V \rightarrow \mathbb{R}$$

Def $\text{support}(f) = \{v \mid f(v) \neq 0\}$

$$\text{supp}_+(f) = \{v \mid f(v) > 0\}$$

$$\text{supp}_-(f) = \{v \mid f(v) < 0\}$$

Def If $Ax = \lambda x$ then x has minimal support

if $\forall y$ st $Ay = \lambda y \Rightarrow \text{support}(y) \not\subset \text{support}(x)$

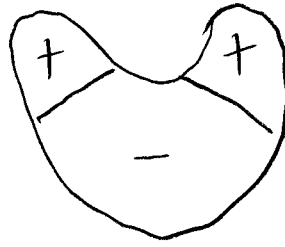
Main Q gen-Lap of G , G connected,

$Qx = \lambda_2 x$, and x has minimal support then

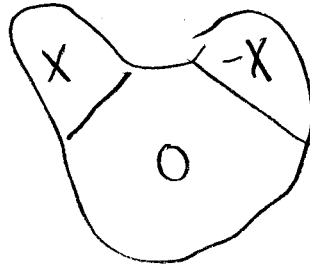
$G(\text{supp}_+(x))$, $G(\text{supp}_-(x))$ are connected.

Picture proof

Suppose x



Consider $\bar{x} \equiv$



adjust $\bar{x}$ s.t. $\bar{x} \perp 1$

Claim $\frac{\bar{x}^T Q \bar{x}}{\bar{x}^T \bar{x}} < \frac{x^T Q x}{x^T x}$ or $\text{support}(\bar{x}) \neq \text{support}(x)$