Iterative Methods for eigenvalues 15-859N

<u>Power Iteration</u> $A = A^T$

<u>Alg</u> Init: $\|V^0\| = 1$ (Pick V^0 at random on unit sphere)

for $K = 1, 2, \cdots$

$w = A\, v^{(K-1)}$ (Sparse matrix-vector product)

$v^{(K)} = w / \|w\|$ (normalize)

$\lambda^{(K)} = v^{(K)} A\, v^{(K)}$ Rayleigh Quotient

<u>Analysis</u>

Init step

<u>Question:</u> Suppose $V = (a_1, \cdots, a_n)$ is random point on unit sphere. How big is a_1?

$$a_1^2 + \cdots + a_n^2 = 1 \Rightarrow E(a_1^2) = 1/n$$

Fact: $\Pr\left[|a_1| \geq \frac{2}{3\sqrt{n}}\right] > .5$

$\lambda_1 > \lambda_2 \geq \lambda_3 \cdots \geq \lambda_n$

Let $q_1, \dots, q_n$ be orthonormal eigenvectors

$$v^{(0)} = a_1 q_1 + \cdots + a_n q_n$$

Consider PI without normalizing

$$\lambda^{(k)} = \frac{v^{(k)T} A v^{(k)}}{v^{(k)T} v^{(k)}} = (*)$$

$$v^{(k)} = a_1 \lambda_1^k q_1 + \cdots + a_n \lambda_n^k q_n$$

$$v^{(k)T} v^{(k)} = a_1^2 \lambda_1^{2k} + \cdots + a_n^2 \lambda_n^{2k}$$

$$v^{(k)T} A v^{(k)} = a_1^2 \lambda_1^{2k+1} + \cdots + a_n^2 \lambda_n^{2k+1}$$

$$* = \frac{\sum a_i^2 \lambda_i^{2k+1}}{\sum a_i^2 \lambda_i^{2k}}$$

$$\lambda_1 - \lambda^{(k)} = \frac{\lambda_1 (\sum a_i^2 \lambda_i^{2k}) - \sum a_i^2 \lambda_i^{2k+1}}{\sum a_i^2 \lambda_i^{2k}}$$

$$= \frac{\left(\sum_{i>1} a_i^2 \lambda_i^{2k}\right)(\lambda_1 - \lambda_i)}{a_1^2 \lambda_1^{2k} + \sum_{i>1} a_i^2 \lambda_i^{2k}}$$

$$\leq \frac{\left(\sum a_i^2 \lambda_2^{2k}\right)(\lambda_1 - \lambda_n)}{a_1^2 \lambda_1^{2k} + \sum a_i^2 \lambda_2^{2k}}$$

$$\leq \frac{(\lambda_1 - \lambda_n)\, \lambda_2^{2k}}{a_1^2 \lambda_1^{2k}} \leq O\left(\left(\frac{\lambda_2}{\lambda_1}\right)^{2k}\right)$$

<u>EG</u> Using Power Iter. for Fiedler vector

$L = L(P_n) \quad \lambda(L) \approx 0, \frac{1}{n^2}, \frac{1}{(n/2)^2}, \frac{1}{(n/3)^2} \cdots 2$

We want $\lambda_2 \approx \frac{1}{n^2}$

$L = D - A$ suppose we power $A \approx 2D - L$

$\lambda_{Max} \approx 2 - \frac{1}{n^2} \qquad \lambda_{next} \approx 2 - \frac{4}{n^2}$

$= \frac{2n^2 - 1}{n^2} \qquad\qquad = \frac{2n^2 - 4}{n^2}$

$\frac{\lambda_{next}}{\lambda_{Max}} \approx \frac{2n^2 - 4}{2n^2 - 1}$ Thus we will need $\approx n^2$ iteration

Inverse Iteration

We have fast solver so lets use it.

Inverse Iteration

Init: Pick random $\|v^0\| = 1 \wedge (v^0)^T \cdot \mathbb{1} = 0$

for $i = 1$ to $k = \frac{c}{\varepsilon} \log\left(\frac{cn}{\varepsilon}\right)$

Solve $A v^{(i)} = v^{(i-1)}$

Return $v^{(k)}$ & $\lambda^{(k)} = \dfrac{v^{(k)T} A v^{(k)}}{v^{(k)T} v^{(k)}}$

Thm For $k = \frac{4}{\varepsilon} \log\left(\frac{18n}{\varepsilon}\right)$ iteration

$$\lambda^{(k)} \leq (1 + \varepsilon/4)\, \lambda_2$$

for $\lambda(A) \equiv 0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$

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pf: Let $r = v^{(0)} = \alpha_2 q_2 + \cdots + \alpha_n q_n$

thus

$$v^{(k)} = \alpha_2 \mu_2^{-k} q_2 + \cdots + \alpha_n \mu_n^{-k} q_n$$

$$v^{(k)T} v^{(k)} = \sum \frac{\alpha_i^2}{\mu_i^{2k}}$$

$$\lambda^{(k)} = \frac{v^{(k)T} A v^{(k)}}{v^{(k)T} v^{(k)}} = \frac{\sum_{i\geq 2} \alpha_i^2/\mu_i^{2k-1}}{\sum_{i\geq 2} \alpha_i^2/\mu_i^{2k}} = (*)$$

Pick $j = \max_j \{\mu_j \leq (1+\varepsilon/8)\mu_2\}$

$$(*) \leq \frac{\sum_{j\geq i\geq 2} \alpha_i^2/\mu_i^{2k-1}}{\sum_{j\geq i\geq 2} \alpha_i^2/\mu_i^{2k}} + \frac{\sum_{i>j} \alpha_i^2/\mu_i^{2k-1}}{\sum_{i\geq 2} \alpha_i^2/\mu_i^{2k}}$$

$$\leq \mu_j + \frac{\sum_{i>j} \alpha_i^2/\mu_i^{2k-1}}{\alpha_2^2/\mu_2^{2k}}$$

$$= \mu_j + \frac{\mu_2}{\alpha_2^2} \sum \alpha_i^2/(\mu_i/\mu_2)^{2k-1}$$

$$\leq u_j + \frac{\mu_2}{\alpha_2^2}\left(\frac{\mu_2}{\mu_{j+1}}\right)^{2k-1}$$

$$\leq \left(1+\frac{\varepsilon}{8}\right)\mu_2 + \frac{\mu_2}{\alpha_2^2}\left(\frac{1}{1+\frac{\varepsilon}{8}}\right)^{2k-1} \qquad (**)$$

Now with prob $\geq 1/2$ we may assume that

$$|\alpha_2| \geq \frac{2}{3\sqrt{n}} \quad \text{or} \quad \alpha_2^2 \geq \frac{4}{9n}$$

Suppose we pick k s.t.

$$\frac{1}{\alpha_2^2}\left(\frac{1}{1+\varepsilon/8}\right)^{2k-1} \leq \left(\frac{4}{9n}\right)^{-1}\left(\frac{1}{1+\frac{\varepsilon}{8}}\right)^{2k-1} \leq \frac{\varepsilon}{8}$$

ie $\left(\frac{1}{1+\frac{\varepsilon}{8}}\right)^{2k-1} \leq \frac{\varepsilon}{18n}$ or $\boxed{\left(1+\frac{\varepsilon}{8}\right)^{2k-1} \geq \frac{18}{\varepsilon}n}$

then

$$(**) \leq \left(1+\frac{\varepsilon}{8}\right)\mu_2 + \frac{\varepsilon}{8}\mu_2 = \left(1+\frac{\varepsilon}{4}\right)\mu_2$$

Thus $\lambda^{(k)} \leq \left(1+\frac{\varepsilon}{4}\right)\mu_2$

We need only solve for k.

$$\boxed{\left(1+\frac{\varepsilon}{8}\right)^{2k-1} \geq \frac{18}{\varepsilon}n}$$

Set $\tau = 2k-1$

$$\left(\left(1+\frac{\varepsilon}{8}\right)^{\frac{8}{\varepsilon}}\right)^{\frac{\varepsilon\tau}{8}} \approx e^{\frac{\varepsilon}{8}\tau} \overset{?}{\geq} \frac{18}{\varepsilon}n$$

ie $$\frac{\varepsilon}{8}\tau \geq \log\left(\frac{18}{\varepsilon}n\right)$$

$$\tau \geq \frac{8}{\varepsilon}\log\left(\frac{18}{\varepsilon}n\right)$$

$$k \gtrsim \frac{4}{\varepsilon}\log\left(\frac{18}{\varepsilon}n\right)$$

Arnoldi Iteration (general matrix)
Lanczos Iter (sym matrix)

<u>Claim</u> Finding roots of poly $\leq_p$ computing eigenvalues

$$P(z) = z^m + a_{m-1} z^{m-1} + \cdots + a_0$$

Consider companion matrix

$$\begin{pmatrix} 0 & & & -a_0 \\ 1 & & 0 & -a_1 \\ & \ddots & & \vdots \\ 0 & & 0 & \vdots \\ & & 1 & -a_{m-1} \end{pmatrix}$$

<u>Note</u> $\det \begin{pmatrix} -z & & & -a_0 \\ 1 & \ddots & & \vdots \\ & \ddots & -z & \vdots \\ & & 1 & -z-a_{m-1} \end{pmatrix} = P(z)$

Iterative methods on Krylov Spaces

Eg Power mag picks $Av^{(k)}$ from

$$\mathcal{K}_n = (v^{(0)}, \ldots, Av^{(n-1)})$$

Question: Can we pick $x \in \mathcal{K}_n$ to max $\frac{x^T A x}{x^T x}$

Question: Can we think of $A : \mathcal{K}_n \to \mathcal{K}_n$?

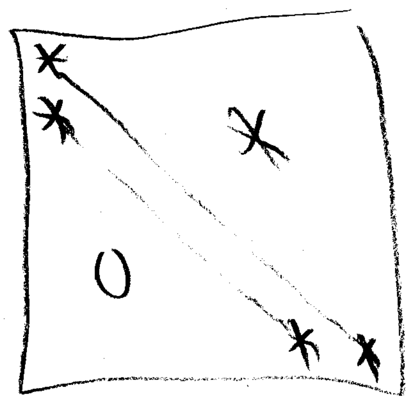
Def $\bar{A}_n(x) = \mathrm{Proj}_{\mathcal{K}_n}(A(x))$

$$Q_n = \begin{pmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{pmatrix}$$ orthnormal basis of $\mathcal{K}_n$

$$\mathrm{Proj}_{K_n} = Q_n Q_n^T$$

Eigenvalues are not rational!

Def H is in Hessenberg form if



Goal: A trans to Hessenberg H with same eigenvr

$$\bar{A}_n = Q_n Q_n^T A$$

$\therefore \bar{A}_n$ is linear trans of $\mathcal{K}_n$ with eigenvalues & vectors.

$y \in \mathcal{K}_n$ written in basis $\begin{pmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{pmatrix}$:

$$A_n y = Q_n^T A Q_n y$$

Goal: Find Q_n st $Q_n^T A Q_n$ is nice.