Differential Equations on Graphs

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First Example: Diffusion

Let $G=(V,E,C)$ undirected $C: E \to \mathbb{R}^+$ (thermal Conductors)

Let $u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ be vector of temperatures

then

$Bu \equiv$ Temp drop across each edge.

$CBu \equiv$ Thermal flow.

$-B^T CBu \equiv$ Change in temp at vertices

Our differential Eq is

$$\frac{\partial u}{\partial t} = -Lu$$

Goal is given $u^{(0)}$ find $u(t)$

note $\dfrac{du}{dt} = \begin{pmatrix} \dfrac{du_1}{dt} \\ \vdots \\ \dfrac{\partial u_n}{dt} \end{pmatrix}$

We use matrix Exponentials

recall $e^x = 1 + x + \dfrac{x^2}{2!} + \cdots$

Def $e^A = I + A + \dfrac{A^2}{2!} + \dfrac{A^3}{3!} + \cdots$ $\begin{pmatrix} \text{sym} \\ A \end{pmatrix}$

If $= U \Lambda U^T$ then $e^A = U e^{\Lambda} U^T$

of $U \begin{pmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{pmatrix} U^T$

Thus $e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots \qquad t \in \mathbb{C}$

then $\frac{d(e^{At})}{dt} = A + \frac{2A^2t}{2!} + \frac{3A^3t^2}{3!} + \cdots$

$$= A(I + At + \cdots)$$

$$= Ae^{At}$$

Claim $u(t) = e^{-Lt}u^{(0)}$ is a solution

1) $u(0) = u^{(0)}$

2) $\frac{d(u(t))}{dt} = \frac{d(e^{-Lt})}{dt}u^{(0)} = -L(e^{-Lt})u^{(0)}$

$$= -L\,u(t)$$

We know $L = U \Lambda U^T$

where $U = \begin{pmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{pmatrix}$ eigenvectors of L

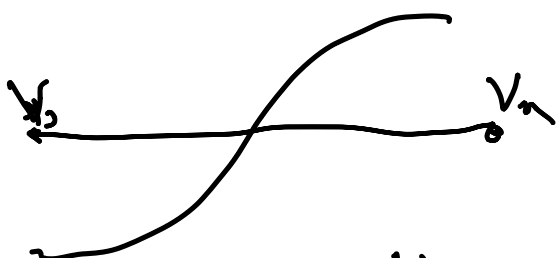
Let $u^{(0)} = c_1 x_1 + \cdots c_n x_n$

then $u(t) = e^{-\lambda_1 t} c_1 x_1 + \cdots + e^{-\lambda_n t} c_n x_n$

EG $G = P_n$

$x_1 = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ no decay

$x_2 =$



$\lambda_2 \approx \frac{1}{n^2}$ $\quad e^{-\lambda_2 t} = e^{-t/n^2}$

halflife $\approx \frac{1}{n^2}$

Springs and Graph Laplacians

Input: Graph of springs (Mattress)

$G = (V, E, k)$

$k_{ij} \equiv$ spring constant for E_{ij}

$m_i \equiv$ mass of vertex V_i

Consider only vertical displacements

eg G is embedded R^2 and

displacement is 3.

Let $u_i \equiv$ displacement of V_i

Goal: Find solutions to Newton Eq.

$F = MA$!

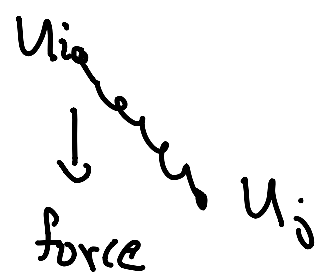
We first must determine F, M, A

F) Find forces for displacement $u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$

Assume linear spring model

force on u_j by spring E_{ij}

$$-(u_i - u_j) k_{ij}$$



If $L = L(G)$ then

Force vector $= -Lu$

M) set $M = \begin{pmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_n \end{pmatrix}$

A) $a = \begin{pmatrix} \frac{d^2 u_1}{dt} \\ \vdots \\ \frac{d^2 (u_n)}{dt} \end{pmatrix} = \frac{d^2 u}{dt^2}$

Thus Newton Eq

$$-Lu = M\left(\frac{d^2 u}{dt^2}\right)$$

Two ways to Solve:

1) Solve for u given initial cond
   Say $u(0)$ & $u'(0)$

2) Find steady state solutions

We will do 2) by guess & check.

Recall: $e^{iwt} = \cos wt + i \sin wt$

$$\frac{de^{iwt}}{dt} = iw\, e^{iwt} = -w \cos wt + iw \cdot \cos wt$$

$$= iw(\cos wt + i \sin wt)$$

$$\frac{d^2 e^{iwt}}{dt^2} = \frac{d(iw\, e^{iwt})}{dt} = i^2 w^2 e^{iwt}$$

$$= -w^2 e^{iwt} = -w^2(\cos wt + i \sin wt)$$

Guess: $u = e^{iwt} X$ for some vector X & some $w \in R$

to show: u works for some (w, X)

$$-L\left(e^{\pm i\omega t}x\right) \overset{?}{=} M\left(-\omega^2 e^{\pm i\omega t}x\right)$$

iff $Lx = \omega^2 Mx$

iff $Lx = \lambda Mx$ for $\lambda \geq 0$

Claim: Eigenvalues of $Lx = \lambda Mx$ are real & non-neg.

<u>pf</u> change of variables

$$y = M^{1/2}x \quad \text{or} \quad x = M^{-1/2}y$$

$Lx = \lambda Mx$ iff $M^{-1/2}LM^{-1/2}y = \lambda y$

The Normalized Laplacian & Spring-Mass Systems

Consider the case

Mass of node = its degree

Then the eigen pair for spring-mass is $LX = \lambda D X$

its sym version is

$D^{-1/2} L D^{-1/2}$ which is the Normalized Laplacian!

Note

$$D^{-1/2} L D^{-1/2} = D^{-1/2}(D-A)D^{-1/2} = I - D^{-1/2} A D^{-1/2}$$