15-859N

11/28/16

Random Spanning Tree

Let $G=(V,E)$

Questions:

1) How many ST does G have?

Def A random ST of G is one picked uniformly from all ST of G.

2) Prob[e ∈ Random ST of G]?

3) Fast algorithm to find a random ST.

$G \equiv$ directed graph

$H \subseteq G$ is <u>divergent ST</u> with root r if

1) r can reach all of $V(G)$ in H.
2) indegree$(v) = 1$ for $v \neq r$.
3) indegree$(r) = 0$

<u>Def</u> D is in-degree matrix of G if

$$D(i,j) = \begin{cases} d_{in}(i) & \text{if } i = j \\ -K = \#\text{edge from } i \text{ to } j & i \neq j \end{cases}$$

$d_{in} \equiv$ in-degree

Note: If G is undirected then $D(G) = L_G$

Let G be a digraph and D its in-degree matrix:

<u>Lemma</u> G is a divergent tree iff

1) $D(i,i) = \begin{cases} 0 & \text{if } i=r \\ 1 & \text{o.w.} \end{cases}$

2) $\det(D_{r,r}) = 1$ where

$D_{r,r}$ is D minus rth row & col.

<u>pf</u> $(\Rightarrow)$ 1) Clear

2) After permuting rows & cols D is upper Δ & $r=1$ thus $\det(D_{r,r})=1$

($\Leftarrow$) Suppose false

G is an digraph with indegree 1 at all nodes but r.

Thus G must contain a cycle C. disjoint from r.

thus D is reducible

$$\begin{bmatrix} * & 0 \\ 0 & C \end{bmatrix}$$

$D(C)$ has zero col sums.

Thus $\det(D(C)) = 0$

$\Rightarrow \det(D_{r,r}) = 0$

Contra!

<u>Thm</u> # divergent ST rooted at $r \equiv \det(D_{r,r})$

pf WLOG assume $r = v_1$

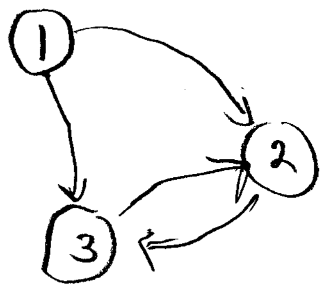
Note: Col sums of D are zero.

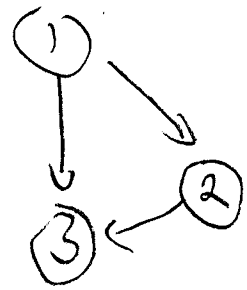
Start by replacing first col with $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$\det(D) = \det(D_{1,1})$.

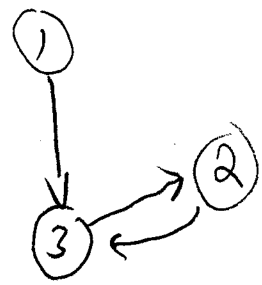
Expand col 2 into subgraphs with fix edge into V_2.

eg





G_1



G_2

$$D = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

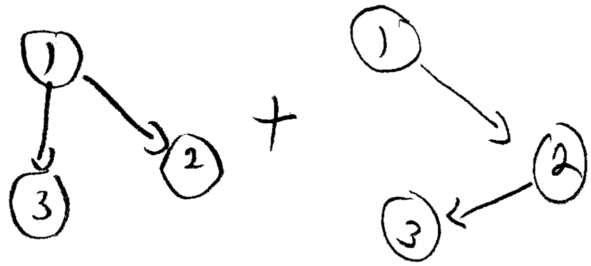
$$|D| = \underbrace{\begin{vmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{vmatrix}}_{G_1} + \underbrace{\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{vmatrix}}_{G_2}$$

For matrix expand next col.

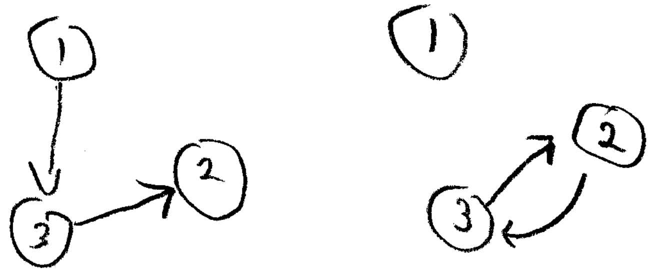
Repeat!

We get exact one matrix per indeg 1 subgraphs.

eg $G_1 \xrightarrow{\text{expand}}$



$G_2 \xrightarrow{\text{expand}}$



Each term is $\begin{cases} 1 & \text{if } \exists \text{ divergent tree} \\ 0 & \text{o.w.} \end{cases}$

$\square$

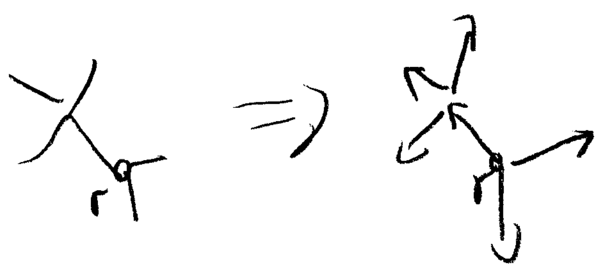
The undirected Case

$G = (V, E)$ undirected

Let $G' = (V, E')$ where each edge $\{i, j\}$ is viewed as (i,j) & (j,i)

<u>Claim</u> $\#ST(G) = \#ST_r(G')/n$

$f: ST(G) \to ST_r(G')$



f is 1-1 & onto

<u>Lemma</u> $\#ST$ of $G = \dfrac{\det(L_{1,1})}{n}$

Let T be the random variable
a random ST of G

Thm $u_e \equiv \text{Prob}[e \in T] = ER_e$

Lemma Matrix Determinate Lemma
Let A be non-singular & u, v col vectors
then $\det(A + uv^T) = \det(A)(1 + v^T A^{-1} u)$

(pf of Lemma)

Consider the identity

$$\begin{pmatrix} A & -u \\ v^T & 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ v^T & 1 \end{pmatrix} \begin{pmatrix} I & -A^{-1}u \\ 0 & 1 + v^T A^{-1} u \end{pmatrix}$$

$$\det\begin{pmatrix} A & -u \\ v^T & 1 \end{pmatrix} = \det\begin{pmatrix} 1 & v^T \\ -u & A \end{pmatrix} = \det\begin{pmatrix} 1 & v^T \\ 0 & A + uv^T \end{pmatrix}$$

Def Let $\tilde{A}$ be A with a row & col removed.

$$\det(\underbrace{RHS}) = \det(A)\cdot\det(\mathbb{1}+v^TA^{-1}u)$$

$(1+v^TA^{-1}u)$ is a scalar thus

$$\det(RHS) = \det(A)(1+v^TA^{-1}u)$$

(pf of Thm)

$$\mu_e = 1-\text{Prob}[e\notin T] = 1-\frac{\det(\widetilde{L_G-\chi_{ij}\chi_{ij}^T})}{\det(\tilde{L}_G)}$$

$$= 1-\frac{\det(\tilde{L}_G-\tilde{\chi}_{ij}\tilde{\chi}_{ij}^T)}{\det(\tilde{L}_G)}$$

$$= 1-\frac{\det(\tilde{L}_G)(1-\tilde{\chi}_{ij}^T\tilde{L}_G\tilde{\chi}_{ij})}{\det(\tilde{L}_G)}$$

$$= \tilde{\chi}_{ij}^T\tilde{L}_G\tilde{\chi}_{ij} = \chi_{ij}^T L_G^+\chi_{ij}$$