15-859N

10/17/16

Graph Cuts & Eigenvalues

2-types of cuts

edge-cuts & vertex-cuts

Today edge-cuts

$G=(V,E,w)$ undirected weighted graph

Def $$\text{Cap}(A,B) = \sum_{\substack{(a,b)\in E \\ a\in A,\, b\in B}} w(a,b)$$

Note: $\text{Cap}(A,A) = \text{Cap}(A)$

$\text{Cap}(A) = \sum_{i\in A} d_i$

Cut measures

1) Quotient cuts or Isoperimetric number

$$\delta(G) = \min_{S\subseteq V} \frac{\text{Cap}(S,\bar{S})}{\min\{|S|,|\bar{S}|\}}$$

2) Sparsest Cut

$$\alpha(G) = \min_{S \subseteq V} \frac{Cap(S,\bar{S})}{|S||\bar{S}|}$$

Note $\frac{n}{2}\alpha(G) \le \beta(G) \le n\,\alpha(G)$

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3) Conductance

$$\Phi(G) = \min_{S \subseteq V} \frac{Cap(S,\bar{S})}{\min\{Cap(S), Cap(\bar{S})\}}$$

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4) Sparest Cut

$$\bar{\alpha}(G) = \min_{S \subseteq V} \frac{Cap(S,\bar{S})}{Cap(S) \cdot Cap(\bar{S})}$$

another view

$$\text{Demand}(V_i, V_j) \equiv d_i \cdot d_j$$

$$\text{Demand}(S,\bar{S}) = \sum_{\substack{s \in S \\ s' \in \bar{S}}} d_s \cdot d_{s'}$$

$$= \left(\sum_{s \in S} d_s\right)\left(\sum_{s' \in \bar{S}} d_{s'}\right) = Cap(S) \cdot Cap(\bar{S})$$

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more generally

$$\bar{\alpha}(G) = \min_{S \subseteq V} \frac{Cap(S, \bar{S})}{Demand(S, \bar{S})}$$

some demand fcn

eg product demand

ie $Demand(v_i, v_j) = P_i \cdot P_j$

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note $\Phi(G)$ is $\approx$ $P_i = \dfrac{d_i}{\sum d_j}$

Let M be a diag matrix of masses.

$$\Phi(G) = \min_{S \subseteq V} \frac{Cap(S, \bar{S})}{\min\{Mass(S), Mass(\bar{S})\}}$$

Main Thms

$L \equiv$ Laplacian of G $\quad \lambda_2 = \lambda_2(L)$

Thm 1 $\lambda_2/2 \leq g(G) \leq \sqrt{2\Delta\lambda_2}$

$\Delta \equiv \max_i \{d_i\}$

$\mathcal{L} \equiv$ Normalized Laplacian of G $\quad \lambda_2 = \lambda_2(\mathcal{L})$

ie $\mathcal{L} = D^{-1/2} L D^{-1/2}$

Thm 2 $\lambda_2/2 \leq \Phi(G) \leq \sqrt{2\lambda_2}$

Thm 3 If $\dfrac{x^T L x}{x^T M x} = \lambda$ then

$$\Phi(G) \leq \sqrt{2 \max_i \left\{\frac{d_i}{m_i}\right\} \lambda}$$

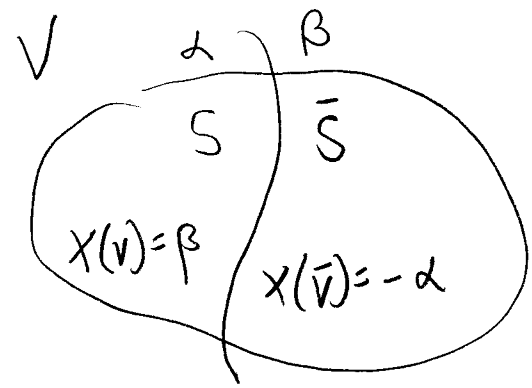
Claim $\lambda_2(L) \le 2 \cdot g(G)$

ie $\exists$ vector x st $x^T \mathbf{1} = 0$ & $\dfrac{x^T L x}{x^T x} \le 2 \cdot g$

pf Let $S \subseteq V$ be st. $|S| \le n/2$

$$\frac{Cap(S,\bar{S})}{|S|} = g$$

set $\alpha = |S|$ & $\beta = |\bar{S}|$



Def $X_i = \begin{cases} \beta & \text{if } i \in S \\ -\alpha & \text{o.w} \end{cases}$

$$X^T \cdot \mathbf{1} = \alpha \cdot \beta - \alpha\beta = 0$$

$$\frac{x^T L x}{x^T x} = \frac{Cap(S,\bar{S})(\alpha+\beta)^2}{\alpha\beta^2 + \beta\alpha^2} = \frac{Cap(S,\bar{S})\, n^2}{\alpha \cdot \beta \cdot n} = \left(\frac{Cap(S,\bar{S})}{\alpha}\right)\left(\frac{n}{\beta}\right)$$

$$\le g \cdot 2$$

Pf of upper bd for Thm 2.

Consider spring-mass version $LX = \lambda MX$

Given $X^T \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0$ Let $\delta = \max_i \left(\frac{d_i}{m_i}\right)$

We show $\Phi(G) \leq \sqrt{2\delta \frac{X^T L X}{X^T M X}}$ (*)

The proof is effective!

ie given X we find a "good" cut.

A "threshold cut"

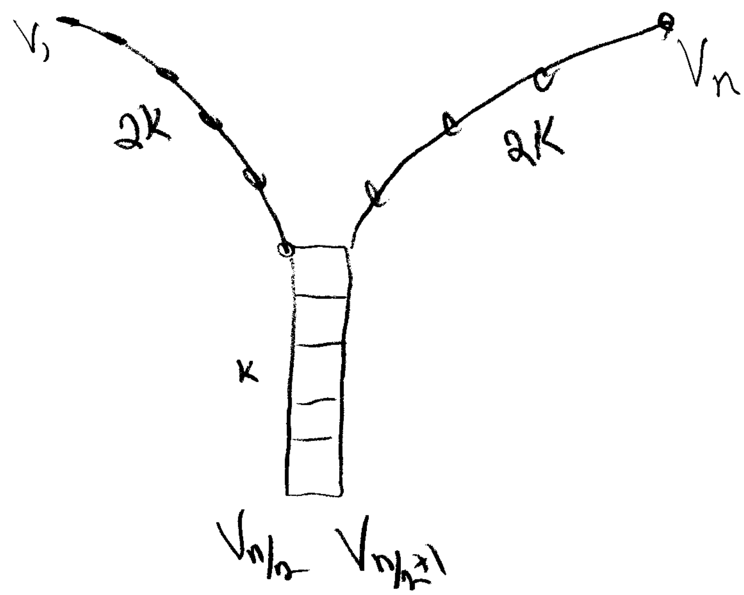
Suppose $f: V \longrightarrow R$

sort V so that $f(v_i) \geq f(v_{i+1})$

Def ith threshold cut $\equiv S_i = \{v_1, \dots, v_i\}$

<u>Note</u> $S_{n/2}$ may not be "good".

Consider Bug-Graph



<u>Claim</u> If $Lf = \lambda_2 f$ then f is odd with respect to its reflection.

The threshold cut for $S_{n/2}$ gives $\frac{K}{3K} \approx 1/3$

$$g \approx \Phi \approx \frac{1}{2K} = \frac{3}{n}$$

Preliminaries $A \equiv$ adj matrix

$$L = D - A \qquad z^T L z = \sum_E w_{ij} (z_i - z_j)^2$$

The sum-Laplacian $P = D + A$

Claim $zPz = \sum_E w_{ij} (z_i + z_j)^2$

Fact $P = 2D - L$

Cauchy-Schwartz

$$(z^T L z)(z^T P z) = \left(\sum w_{ij} (z_i - z_j)^2\right)\left(\sum w_{ij} (z_i + z_j)^2\right)$$

$$\left(\sum \left(\sqrt{w_{ij}}\, |z_i - z_j|\right)^2\right)\left(\sum \left(\sqrt{w_{ij}}\, |z_i + z_j|\right)^2\right)$$

$$\geq \left(\sum w_{ij} \left| z_i^2 - z_j^2 \right|\right)^2$$

Here $a = (\cdots, \sqrt{w_{ij}}\, |z_i - z_j|, \cdots)$

$b = (\cdots, \sqrt{w_{ij}}\, |z_i + z_j|, \cdots)$

<u>The Proof</u> Let $z \in \mathbb{R}^n$ st

$z^T M \bar{1} = 0$ & $z_1 \geq z_2 \cdots \geq z_n$

<u>Def</u> $\Phi_i = \dfrac{\mathrm{Cut}(S_i, \bar{S}_i)}{\min\{\mathrm{Mas}(S_i), \mathrm{Mas}(\bar{S}_i)\}}$

<u>to show</u> $\exists i \quad \Phi_i \leq \sqrt{2\delta \dfrac{z^T L z}{z^T M x}}$

We modify z & G only decreasing

$\dfrac{z^T L z}{z^T M z}$ & not changing Φ_i

<u>Def</u> $\beta = \arg\max_i \{ \mathrm{Mas}(\bar{S}_i) \leq \mathrm{Mas}(S_i) \}$

two modifications to z & G.

1) Shifting z st $z_\beta = 0$

2) $\forall (i,j) \in E$ st $z_i \cdot z_j < 0$ pinch (i,j) at β.

1) Shifting z: $y = z + \alpha \vec{1}$

Claim $\dfrac{y^T L y}{y^T M y} \le \dfrac{z^T L z}{z^T M z}$

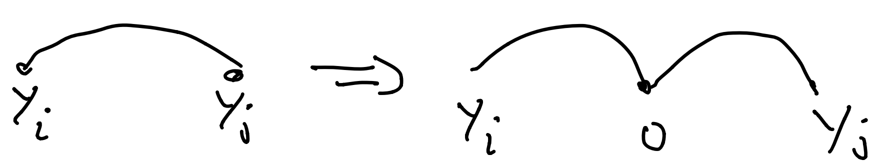
$$\frac{y^T L y}{y^T M y} = \frac{(z + \alpha\vec{1})^T L (z + \alpha \vec{1})}{(z + \alpha\vec{1})^T M (z + \alpha\vec{1})}$$

$$= \frac{z^T L z}{z^T M z + 2\alpha z^T M \vec{1} + \alpha^2 \vec{1}^T M \vec{1})} \le \frac{z^T L z}{z^T M z}$$

since $z^T M \vec{1} = 0$

<u>Pinching an edge at zero</u>

For each edge st $y_i < 0 < y_j$



This give Laplacian L'

<u>Claim</u> $Y^T L' Y \le Z^T L Z$

each term $(z_i - z_j)^2$ is replaced with

$$(z_i - 0)^2 + (0 - z_j)^2$$

$$(z_i - z_j)^2 = z_i - 2 z_i z_j + z_j^2 > z_i^2 + z_j^2$$

since $z_i z_j < 0$

<u>Note</u> The Masses are unchanged.

the degree at β may increase

ie $D' \geq D$

For sum-Laplacians

$$\frac{Y^T P' Y}{Y^T M Y} = \frac{Y^T (2D' - L') Y}{Y^T M Y} \le \frac{2 Y^T D' Y}{Y^T M Y} \overset{Y_\beta = 0}{=} 2 \frac{Y^T D Y}{Y^T M Y}$$

$$\le 2 \max_i \left(\frac{d_i}{m_i} \right) = 2\delta .$$

$$2\delta \frac{z^T L z}{z^T M z} \cdot (y^T M y)^2 \geq (y^T P' y) \left(\frac{z^T L z}{z M z}\right) y^T M y$$

$$\geq (y^T P' y)(y^T L' y) \geq \left(\sum w_{ij} \left| y_i^2 - y_j^2 \right| \right)^2$$

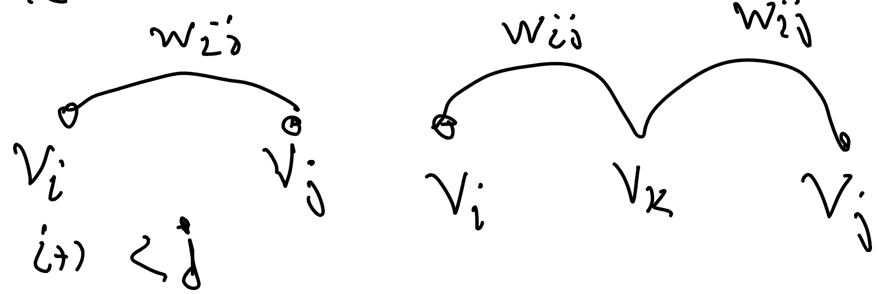
<u>thus</u>

$$\sqrt{2\delta \frac{z^T L z}{z^T M z}}\, (y^T M y) \geq \sum_{i<j\leq\beta} w_{ij} (y_i^2 - y_j^2)$$

$$+ \sum_{j>i\geq\beta} w_{ij} (y_i^2 - y_j^2)$$

Claim WLOG G is a weighted path graph

ie



Let new edge weights w_i from V_i to V_{i+1}

Let $\Phi_i = \frac{w_i}{Mas(S_i)}$ & $\Phi^*_{Left} = \min_{i \leq \beta} \Phi_i$ & $Mas(V_0) = 0$

$$\sum_{i=1}^{\beta-1} w_i (y_i^2 - y_{i+1}^2) = \sum_{i=1}^{\beta-1} \Phi_i \, Mas(S_i)(y_i^2 - y_{i+1}^2)$$

$$\geq \Phi^*_{Left} \sum_{i=1}^{\beta-1} Mas(S_i)(y_i^2 - y_{i+1}^2)$$

$$= \Phi^*_{Left} \sum_{i=1}^{\beta-1} (Mas(S_i) - Mas(S_{i-1})) y_i^2$$

$$= \Phi^*_{Left} \sum_{i=1}^{\beta-1} m_i y_i^2 = \Phi^*_{Left} \, y_{left}^T M y_{left}$$

$$\sqrt{2\delta \frac{z^T L z}{z^T M z}}\,(y^T D y) \geq \Phi^*_{left}\, y_{left}^T M y_{left} + \Phi^*_{right}\, y_{right}^T M y_{right}$$

$$\geq \Phi^* (y_l^T M y_l + y_r^T M y_r)$$

$$= \Phi^* y^T M y$$

Thus

$$\sqrt{2\delta \frac{z^T L z}{z^T M z}} \geq \Phi^*$$