Bounding Eigenvalues

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Thm (Courant-Fischer) $A^{n\times n}$ sym with eigens $\mu_1 \geq \cdots \geq \mu_n$ then

$$\mu_k = \max_{\substack{S \subseteq R^n \\ \dim(S)=k}} \min_{x \in S} \frac{x^T A x}{x^T x}$$

$$= \min_{\substack{T \subseteq R^n \\ \dim(T)=n-k+1}} \max_{x \in T} \frac{x^T A x}{x^T x}$$

View CF as a two person game.

Let $x_1, \cdots x_n$ be the eigenvectors.

<u>pf</u> Case max-min for S

($\geq$)

pick $S = \langle x_1, \cdots, x_k \rangle$ & $x \in S$

$$x = \sum c_i x_i$$

$$\therefore \quad \frac{x^T A x}{x^T x} = \frac{\sum c_i c_i^2}{\sum c_i^2} \geq \frac{\sum \mu_k c_k^2}{\sum c_i^2} = \mu_k$$

($\leq$) to show $\forall S$ $\dim(S) = k$

$$\min_{x \in S} \frac{x^T A x}{x^T x} \leq \mu_k$$

Pf Let $T = \langle x_k, \cdots, x_1 \rangle$

Thus $\dim(T) = n-k+1$

$\therefore T \cap S \neq \emptyset$

$$\min_{x \in S} \frac{x^T A x}{x^T x} \leq \min_{x \in S \cup T} \frac{x^T A x}{x^T x}$$

$\therefore x \in T \quad x = \sum_{i=k}^{n} c_i x_i$

$$\frac{x^T A x}{x^T x} = \frac{\sum \mu_i c_i^2}{\sum c_i^2} \leq \frac{\sum \mu_k c_i^2}{\sum c_i^2} = \mu_k$$

$\square$

Let A adj matrix & D degree

Let $\mu_1 \geq \cdots \geq \mu_n \in \lambda(A)$

d_{max}, d_{ave} be max & average degree

<u>Lemma</u> $d_{ave} \overset{(a)}{\leq} \mu_1 \overset{(b)}{\leq} d_{max}$

<u>pf</u> (a)

$$\mu_1 = \max_x \frac{x^T A x}{x^T x} \geq \frac{\mathbb{1}^T A \mathbb{1}}{\mathbb{1}^T \mathbb{1}} = \frac{2m}{n} = d_{ave}$$

(b)

Let $Ax = \mu_1 x$ & $x_v \geq x_u \ \forall u \in V$

$$\mu_1 = \frac{(Ax)_v}{x_v} = \frac{\sum x_u A_{vu}}{x_v} \leq \sum A_{vu} = d_{max}$$

<u>Lemma</u> If G is connected then G is d-reg iff $\mu_1 = d = d_{max}$

$(\Rightarrow)$ $A\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix} = d\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}$

$(\Leftarrow)$ $Ax = dx$ $\quad\quad \max(x) = x_v$

$$d x_v = \sum_{(v,u)\in E} x_u \Rightarrow x_u = x_v$$

connected $\Rightarrow$ $x_u = x_v \ \forall u, v$

<u>Lemma</u> G is graph $L = L(G)$

with eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$

then $\lambda_n \leq 2 d_{max}$

pf $L = D - A$

$$\lambda_n \leq \lambda_{max}(D) - \lambda_{min}(A)$$

$$\leq d_{max} - (-d_{max}) = 2 d_{max}$$

A, B $n \times n$ Sym

New Notation: $A \succeq 0$ if $\forall x$ $x^T A x \geq 0$

In general: $A \preceq B$ if $\forall x$ $x^T A x \leq x^T B x$

ie. $B - A \succeq 0$

Note: $A \preceq B$ & $B \preceq C$ then $A \preceq C$

$A \preceq B \Rightarrow A + C \preceq B + C$

Graph G & H $\quad G \preceq H$ if $L_G \preceq L_H$

Thus $G \preceq G \cup H$

<u>Thm</u> $A \preceq B \Rightarrow \lambda_k(A) \le \lambda_k(B)$

Pf

$$\lambda_k(B) = \min_{\dim(T)=n-k+1} \max_{x \in T} \frac{x^T B x}{x^T x}$$

$$\ge \min_{\dim(T)=n-k+1} \max_{x \in T} \frac{x^T A x}{x^T x}$$

$$= \lambda_k(A)$$

Pesudo-Inverse

$a \in R$

Def $a^+ = \begin{cases} 0 & \text{if } a = 0 \\ a^{-1} & \text{o.w} \end{cases}$

D is diagonal.

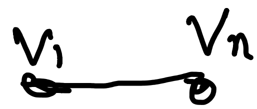
Def $(D^+)_{ii} = D^+_{ii}$

A is sym $\Rightarrow U^T \Lambda U$

Def $A^+ = U^T \Lambda^+ U$

$G = (V, E, w) \quad V = \{v_1, \dots, v_n\}$

$G_{1n} = (V, E = \{(1,n)\})$



$ER_{ij} \equiv$ effective resistance in G.