Arnoldi & Lanczos Revisited

15-859N
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Recall $H^{m\times m} = \begin{pmatrix} * & & * \\ * & \ddots & \\ 0 & \ddots & * \; * \end{pmatrix}$ $\quad \bar{H}_n = \begin{pmatrix} | & & | & \\ h_1 & \cdots & h_n & 0 \\ | & & | & \end{pmatrix}$

$$= \left(\begin{array}{c|c} H_n & 0 \\ \hline 0 \;\; * & 0 \end{array}\right)$$

$Q = \begin{pmatrix} | & & | \\ q_1 & \cdots & q_m \\ | & & | \end{pmatrix} \quad Q_n = \begin{pmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{pmatrix} \quad Q^T Q = I$

Arnoldi Iter gave $AQ = QH$

$\mathcal{K}_n = \langle b, Ax, \ldots, A^{n-1}b \rangle = \langle q_1, \ldots, q_n \rangle$

$\mathbb{P}_n = \{\text{monic poly of degree } n\} \quad c_n = 1$

Arnoldi / Lanczos Approx Prob

Find $p \in \mathbb{P}_n$ to min $\|p(A)b\|$

Thm If $\mathcal{K}_n$ not of full rank then

1) p^n is unique 2) $p^n = \text{char}(H_n)$

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1) Uniqueness $p \in \mathbb{P}_n$ & $\|p(A)b\|$ min

a) p monic $\Rightarrow$ $p(A)b = A^n b - Q_n y$ $y \in \mathbb{R}^n$

where $Q_n y \in \mathcal{K}_n$

b) $\|p(A)b\|$ min $\Rightarrow$ $Q_n y \equiv \text{proj}(A^n y)$ onto $\mathcal{K}_n$

thus $p(A)b \perp \mathcal{K}_n$ ie

$Q_n^T p(A) b = 0$

Thus p is unique

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2) Claim $g = \text{char}(H_n) \Rightarrow Q_n^T g(A) b = 0$

and thus g is minimizer $\|p(A)b\|$

pf of 2)

$$Q_n^T q(A) b = 0$$

iff $Q_n^T Q\, q(H)\, Q^T b = 0$ $\qquad A = QHQ^T$

iff $Q_n^T Q\, q(H) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0$ $\qquad b = c\, q_1$

iff $Q_n^T Q\, q(\bar{H}_n) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0$ $\qquad \deg(q) \leq n$

iff $q(H_n) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0$ $\qquad Q_n^T Q = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ 0 & & 1 & \end{pmatrix}$

<u>thus</u> $q(H_n) \neq 0 \Rightarrow Q_n^T q(A) b = 0$