15-859N Spectral Graph Theory and The Laplacian Paradigm, Fall 2016

Homework 1 **Due: Monday Nov 7**

Gary Miller TAs: Goran, and Ilqar

Instructions. Collaboration is permitted in groups of size at most three. You must write the names of your collaborators on your solutions and must write your solutions alone.

Question	Points	Score
1	25	
Total:	25	

(25) 1. **Leverage Scores and Resistors**

Let $A^{m \times n}$ be a matrix of rank n.

Definition 1 *The* **Leverage Score** $\sigma(A, a)$ *where a is a column vector of size n is $a^T(A^TA)^\dagger a$ where $\dagger$ is the pseudoinverse. If a_i is the ith row of A the $\sigma_i(A) = \sigma(A, a_i^T)$.*

The goal of this problem is to determine what if any of the properties of effective resistance carry over the leverage scores.

1. Show that the leverage score of a row vector with itself is one.
2. Show that the column space $Col(A)$ of A and the left null space $Null_L(A)$ of A form an orthogonal bases of R^m. We will think of these vectors as the flows of A.
3. We next prove a generalization of Foster's Theorem. Show that the sum of the row leverage scores of A is $rank(A)$.
4. We next prove a generalization of Thomsons Principle. Suppose that x is a solution to the system $A^TAx = b$ were b is in the column space of A^T. Show that the flow $f = Ax$ is the unique minimum energy flow such that $A^Tf = b$. We define the energy of f to be f^Tf.
5. We next prove a generalization of Rayleigh's Monotonicity Law. If we increase a row of A by scaling it by $1 + c$ for $c > 0$ or add a new row then no leverage score except the changed one will increase.
6. We next prove a generalization of Spielman-Srivastava Graph Sparsification Theorem. We say that $A^TA \approx_\epsilon B^TB$ if $(1-\epsilon)A^TA \preccurlyeq B^TB \preccurlyeq (1+\epsilon)A^TA$. Prove that that there exist a matrix QB where B is a subset of m' rows of A and Q is a nonnegative diagonal matrix where $m' = cn \log n$ for some constant c and $A^TA \approx_\epsilon B^TB$.

7. Prove a variant of the fact that conductors add when placed in parallel. In particular prove a relationship between $\sigma(a, a)$, $\sigma(A, a)$, and $\sigma(\bar{A}, a)$ where $\bar{A}$ is the matrix A with row a appended.

   Hint: Consider the Sherman-Morrison formula.

   Can you find a more general formula?

Research questions:

1. We also showed that the effective resistance forms a metric over the **vertices**. Thus in our generalization we should be looking for a metric on the columns of A. Suppose we define the score between two columns as $D_{ij} = \chi_{ij}^T (A^T A)^{-1} \chi_{ij}$ where χ_{ij} is the column vector with a 1 and -1 at i and j respectively. Does our score say anything interesting about the relationship of two columns? Is D_{ij} a metric on the columns of A.
2. Is there theory of random walks for leverage scores, either on the clumns or rows of A?
3. There in such a theory does commute time make sense and is related to leverage score?