Duality

Primal: min $c^T x$

$$Ax = b$$
$$x \geq 0$$

Dual max $b^T y$

$$A^T y \leq c$$

Example: P: minimise $3x_1 + 2x_2 + 4x_3$

s.t. $2x_1 + x_2 + x_3 = 7$

$5x_1 + x_2 - x_3 = 9$

$x_1, x_2, x_3 \geq 0$

D: maximise $7y_1 + 9y_2$

s.t. $2y_1 + 5y_2 \leq 3$

$y_1 + y_2 \leq 2$

$y_1 - y_2 \leq 4$

(ii) Primal Value $c_{B^*}^T B^{*-1} b$

$$\parallel$$

Dual Value $b^T y^* = b^T B^{*-T} c_{B^*}$

Example

P: minimize $-3x_1 - x_2 + x_3$

s.t.

$$\begin{aligned} x_1 + x_2 + x_3 &= 10 \\ 2x_1 - x_2 - x_4 &= 2 \\ x_1 - 2x_2 + x_3 + x_5 &= 6 \\ x_1, \dots x_6 &\geq 0 \end{aligned}$$

D: maximize $10y_1 + 2y_2 + 6y_3$

s.t.

$$\begin{aligned} y_1 + 2y_2 + y_3 &\leq -3 \\ y_1 - y_2 - 2y_3 &\leq -1 \\ y_1 + y_3 &\leq 1 \\ -y_2 &\leq 0 \\ y_3 &\leq 0 \end{aligned}$$

Dual of dual = Primal!

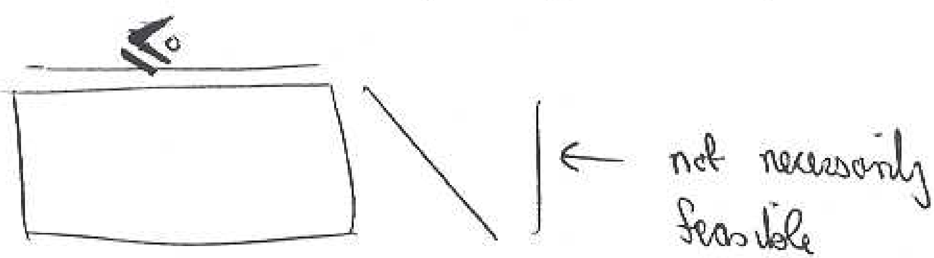
$$B^* = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & -1 \\ 1 & -2 & 0 \end{bmatrix} \end{array} \qquad c_{B^*} = [-3, -1, 0]^T$$

$$B^{*-1} = \begin{bmatrix} 2/3 & 0 & 1/3 \\ 1/3 & 0 & -1/3 \\ 1 & -1 & 1 \end{bmatrix} \qquad y^* = [-7/3, 0, -2/3]^T$$

$$\text{Value} = -\frac{82}{3}$$

Dual Feasible Basis (Minimisation)

B is dual feasible if $c_j - c_B^T B^{-1} a_j \geq 0 \;\; \forall j$



Equivalently: $(c_B^T B^{-1})^T$ is feasible for dual

Other forms of Primal $\to$ Standard Form $\to$

Dualise $\to$ Simplify.

Example.

<u>Primal v Dual</u>

x is feasible for Primal and y is feasible for Dual

$$\Rightarrow \quad c^T x \geq b^T y$$

$$\begin{aligned} c^T x - b^T y &= x^T c - x^T A^T y \\ &= x^T (c - A^T y) \\ &\geq 0 . \end{aligned}$$

So if x^*, y^* feasible for Primal, Dual resp. and $c^T x^* = b^T y^*$ then both optimal.

Suppose B^* is optimal basis for Primal

$$y^* = B^{*-T} c_{B^*} \quad \text{solves Dual}$$

(i) $\quad c_j - c_{B^*}^T B^{*-1} a_j \geq 0 \qquad$ optimality of B^*

or $\quad c - y^{*T} A \geq 0 \qquad$ y^* feasible

Dual value $= \underline{b}^T y^* = \underline{b}^T B^{*-T} \underline{c}_B^* = \underline{c}_B^{*T} B^{*-1} \underline{b}$

$=$ Primal value

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Complimentary Slackness

Consider the (Primal) Linear Program

$$\begin{aligned}
&\text{minimize} \quad c_1x_1 + c_2x_2 + c_3x_3 \\
&\text{subject to} \quad A_{11}x_1 + A_{12}x_2 + A_{13}x_3 \geq b_1 && (R_1) \\
&\qquad A_{21}x_1 + A_{22}x_2 + A_{23}x_3 \leq b_2 && (R_2) \\
&\qquad A_{31}x_1 + A_{32}x_2 + A_{33}x_3 = b_3 && (R_3) \\
&\qquad\quad x_1 \geq 0, \quad x_2 \leq 0, \quad x_3 \text{ unconstrained}
\end{aligned}$$

Dual

$$\begin{aligned}
&\text{maximize} \quad b_1y_1 + b_2y_2 + b_3y_3 \\
&\text{subject to} \quad A_{11}y_1 + A_{21}y_2 + A_{31}y_3 \leq c_1 && (DR_1) \\
&\qquad A_{12}y_1 + A_{22}y_2 + A_{32}y_3 \geq c_2 && (DR_2) \\
&\qquad A_{13}y_1 + A_{23}y_2 + A_{33}y_3 = c_1 && (DR_3) \\
&\qquad\quad y_1 \geq 0, \quad y_2 \leq 0, \ y_3 \text{ unconstrained}
\end{aligned}$$

Observe the relationship

Primal (Min)		Dual (Max)
Variable		Constraint
$x_1 \geq 0$	————	$DR_1 \leq$
$x_2 \leq 0$	————	$DR_2 \geq$
x_3 unconstrained	————	$DR_3 =$

Constraint	Variable
$R_1 \geq$	$y_1 \geq 0$
$R_2 \leq$	$y_2 \leq 0$
$R_3 =$	y_3 unconstrained

Observe that if $\underline{x}, \underline{y}$ are feasible for the primal, dual respectively then (check!)

$$(c_1x_1 + c_2x_2 + c_3x_3) - (b_1y_1 + b_2y_2 + b_3y_3) =$$

$$\sum_{i=1}^{3} y_i(A_{i1}x_1 + A_{i2}x_2 + A_{i3}x_3 - b_i)$$

$$+ \sum_{j=1}^{3} x_j(C_j - A_{1j}y_1 - A_{2j}y_2 - A_{3j}y_j) \geq 0.$$

So

$cx^* = by^*$ and both are optimal if

$x_j^* \neq 0 \Rightarrow DR_j$ is tight with y^*

$y_i^* \neq 0 \Rightarrow R_i$ is tight with x^*

(A constraint is tight if it is satisfied with equality)