Chapter 3

Some Harder Games and How to Make Them Easier

Our life is frittered away by detail... Simplify, simplify.
Henry David Thoreau. Walden.

Figure 1. A Well Advanced Game of Poker-Nim.

POKER-NIM

This game is played with heaps of Poker-chips. Just as in ordinary Nim, either player may reduce the size of any heap by removing some of the chips. But now we allow a player the alternative move of increasing the size of some heap by adding to it some of the chips he acquired in earlier moves. These two kinds of moves are the only ones allowed.

Let's suppose there are three heaps, of sizes 3,4,6 as in Fig. 1, and that the game has been going on for some time, so that both players have accumulated substantial reserves of chips. It's Left's turn to move, and he moves to 2,4,6 since he remembers from Chapter 2 that this is a good move in ordinary Nim. But now Right adds 50 chips to the 4 heap, making the position into 2,54,6, which is well beyond those discussed in Chapter 2.
This seems somewhat disconcerting, especially since Right has plenty more chips at his disposal, and doesn’t seem too scared of using them to complicate the position. What does Left do? After a moment’s thought, he just removes the 50 chips Right has just added and waits for Right’s reply. If Right adds 1000 chips to one of the heaps, Left will remove them and restore the position to 2, 4, 6 once again. Sooner or later, Right must reduce one of the three heaps (since otherwise he’ll run out of chips no matter how many he has), and then Left can reply with the appropriate Nim-move.

So whoever can win a position in ordinary Nim can still win in Poker-Nim, no matter how many chips his opponent has accumulated. He replies to the opponent’s reducing moves just as he would in ordinary Nim, and reverses the effect of any increasing move by using a reducing move to restore the heap to the same size again. The new moves in Poker-Nim can only postpone defeat, not avoid it indefinitely. Since the effect of any of the new moves can be immediately reversed by the other player, we call them reversible moves.

NORTHCOTT’S GAME

The same sort of thing happens in other games, often in better disguise. Northcott’s game is played on a checkerboard which has one black and one white piece on each row, as in Fig. 2. You may move any piece of your own color to any other empty square in the same row, provided you do not jump over your opponent’s piece in that row. If you can’t move (because all your pieces are trapped at the side of the board by your opponent’s), you lose.

This can seem an aimless game if you don’t see the point, and indeed it usually goes on forever if it is played badly. But when you realize that it’s only Nim in disguise once more, you’ll soon be able to beat anybody pretty quickly. To the left of the board in Fig. 2 we have shown the numbers of spaces between the two pieces in each row. When someone moves, just one of these numbers will be changed, and might be either increased or decreased, according as the move was retreating or advancing. But just as in Poker-Nim, any moves increasing one of the numbers can be reversed by the next player, and so are not much use.

Who wins in Fig. 2? We can see the zero-position 2, 4, 6 among the numbers shown, and of course the two n (1, the only other reduce this 5 to should reduce it opponent, never

It should not Fig. 2 instead of White) or 0 by 5 not always reply retreating moves

BOGUS NIM

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We summarize

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course the two numbers 3 together form another zero. Neglecting the two rows that are already 0, the only other number is 5, and we maintain that the first player can win by moving so as to reduce this 5 to 0. Whenever the other player enlarges some gap by retreating, the first player should reduce it again by the same extent. In fact the winner should always advance on his opponent, never retreat.

It should not be thought that the moves we advise here are the only good ones. For example, from Fig. 2 instead of reducing 5 to 0, we could replace 6 by 3, 4 by 1 or even 3 by 6 in the second row (for White) or 0 by 5 in the last row (for Black). In fact it will help to avoid revealing the strategy if you do not always reply to a retreating move by the corresponding advance—for similar reasons occasional retreating moves might be desirable.

BOGUS NIM-HEAPS AND THE MEX RULE

Consider the impartial game


This is a new kind of Nim-heap from which either player can move to a heap of size 0, 1, 2, 5, 6, or 9. In other words, we can regard it as a rather peculiar Nim-heap of size 3 (the first missing number), from which, as well as the usual moves to heaps of sizes 0 or 1 or 2, we are allowed to move to a heap of size 5 or 6 or 9. However, the Poker-Nim argument shows that this extra freedom is in fact of no use whatever.

To be more precise, suppose some player has a winning strategy in the game *3 + H + K + .... Then in the same circumstances, he has one in G + H + K + .... When his strategy calls for a move from any of *3, H, K, ...., that move is still available, and he need not use the new permitted moves from G to *5, *6, or *9. If his opponent tries to do so, he can immediately reverse the effect of this move by moving back to *3 (since 5, 6, and 9 are all greater than 3), and revert to the original strategy. So G can be replaced by *3 without affecting either player's chances.

The same argument shows that any game of the form

$$G = \{a, b, c, ...|a, b, c, ...\},$$

in which the same numbers appear on both sides, is really a Nim-heap in disguise. For if m is the least number from 0,1,2,3,... that does not appear among the numbers a,b,c,..., then either player can still make from G any of the moves to *0,*1,*2,...,*m−1 that he could make from *m. If his opponent makes any other move from G, it must be to some *n for which n > m, and can be reversed by moving back from *n to *m. So G is really just a bogus Nim-heap *m.

We summarize:

If Left and Right have exactly the same options from G, all of which are Nim-heaps *a, *b, *c,...., then G can itself be regarded as a Nim-heap, *m, where m is the least number 0 or 1 or 2 or .... that is not among the numbers a, b, c,....

THE MINIMAL-EXCLUDED (MEX) RULE

This minimal-excluded number is called the mex of the numbers a, b, c,....
THE SPRAGUE-GRUNDY THEORY FOR IMPARTIAL GAMES

The above result enables us to show that every impartial game can be regarded as a bogus Nim-heap. For suppose we have an impartial game

\[ G = \{A,B,C,\ldots;A,B,C,\ldots\}. \]

Then \(A, B, C, \ldots\) are simpler impartial games, and therefore we can suppose they have already been shown to be equivalent to Nim-heaps \(a_1, b_1, c_1, \ldots\). But in this case \(G\) can be thought of as the Nim-heap \(m\) defined above. This gives us

THE BOGUS NIM-HEAP PRINCIPLE

Every impartial game is just a bogus Nim-heap (that is, a Nim-heap with reversible moves added from some positions). The Mex Rule gives the size of the heap for \(G\) as the least possible number that is not the size of any of the heaps corresponding to the options of \(G\).

This principle was discovered independently by R.P. Sprague in 1936 and P.M. Grundy in 1939, although they did not state it in quite this way. This means that provided we can play the game of Nim, we can play any other impartial game given only a “dictionary” saying which numbers (i.e., Nim-heaps) correspond to the positions of that game. Here’s a game played with a White Knight that gives a simple example of this dictionary method.

![Figure 3. The White Knight and his Baggage.](image)

THE WHITE K

Fig. 3. You may recollect that they are boxes. The game ends when all the boxes are empty. The Mex Rule gives the size of the heap for \(G\) as the least possible number that is not the size of any of the heaps corresponding to the options of \(G\).

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The whole game Table 1 shows why the four places he can move in Fig. 3 is to...
THE WHITE KNIGHT

has, from any position on the chessboard, the moves shown in Fig. 3. You may recall that he was in the habit of losing his belongings. Alice has kindly boxed them up and the boxes now form the Nim-heap to the right of the figure. Now consider the game in which you can either move the Knight to one of the four places shown or steal some of the boxes. The game ends only when the Knight is on one of the four home squares and all the boxes have gone.

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Table 1. Nimbers for the White Knight.

The whole game is the result of adding a Nim-heap *6 to a game with only the Knight. Table 1 shows which nimbers correspond to the game with the Knight in various positions. Let's find the value of the Knight on d7 as in Fig. 3, assuming we already know the values of the four places he can move to. Figure 4 shows that these places can be thought of as bogus Nim-heaps of sizes

0, 3, 0, 1 (mex = 2)

and so the present position corresponds to a bogus Nim-heap of size 2, value *2. So the good move in Fig. 3 is to steal all but two of the boxes.
Figure 4. What the White Knight Moves are Worth.

**Adding Nimbers**

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Table 2. A Nim-Addition Table.
We saw in Chapter 2 that a Nim-heap of size 2 together with one of size 3 is equivalent to one of size 1. We now see that this was no accident, for the sum of any two Nim-heaps \( a \) and \( b \) is an impartial game, and so is equivalent to some other Nim-heap \( c \). The number \( c \) is called the nim-sum of \( a \) and \( b \), and written \( a \oplus b \). How can we work out nim-sums in general?

The options from \( a \oplus b \) are all the positions of the form \( a' \oplus b \) or \( a \oplus b' \) in which \( a' \) denotes any number (from 0,1,2,…) less than \( a \), and \( b' \) any number (from 0,1,2,…) again less than \( b \). So \( a \oplus b \) is the least number 0,1,2,… not of either of the forms

\[
a' + b, \quad a + b' \quad (a' < a, \ b' < b)
\]

Table 2 was computed using this rule. For example the entry \( 6 \oplus 3 \) was computed as follows. The earlier entries 3,2,1,0,7,6 in column 3 correspond to the options \( *6 + *3 \) (where \( * \) means one of 0,1,2,3,4,5) and the earlier entries 6,7,4 in row 6 correspond to options \( *6 + *3' \) (3' means 0,1 or 2). The least number not observed earlier in either row or column is 5, so \( 6 \oplus 3 = 5 \), i.e. \( *6 + *3 = *5 \). It might help you to follow how the table is computed if you look at the game in which our White Knight is replaced by a White Rook which can only move North or West.

You’ll find a general Nim-Addition Rule in the Extras, and will have many opportunities to apply it; for example, in Chapters 4, 12, 14 and 15.

**WYT QUEENS**

\[
\text{Figure 5. How the Wyt Queens Move.}
\]
In the game of Wyt Queens any number of Queens can be on the same square and each player, when it is his turn to move, can move any single Queen an arbitrary distance North, West or North-West as indicated, even jumping over other Queens.

Because the Queens move independently, we can regard the whole game as the sum of smaller ones with just one Queen. The various Queens on the board will therefore correspond to Nim-heaps $a,b,c,...$ which we can add using the Nim-Addition Rule. Try computing the nimber dictionary for this game—when you get tired you can look in the Extras for more information.

The one-Queen game is a transformation of Wythoff's Game (1905) played with two heaps in which the move is to reduce either heap by any amount, or both heaps by the same amount. We'll meet Wyt Queens again in Chapters 12 and 13.

**REVERSIBLE MOVES IN GENERAL GAMES**

What does it mean for some move to be reversible in an arbitrary game $G$? We shall suppose Right's move to $D$ is reversible in the game

$$G = \{A,B,C,...\mid D,E,F,...\}.$$ 

This will mean that there is some move for Left from $D$ to a Left option $D^L$ which is at least as good for Left as $G$ was, i.e. $D^L \geq G$. Then if ever Right moves from $G$ to $D$, Left can at least reverse the effect by moving back from $D$ to $D^L$, and might even improve his position by doing so. We shall suppose that $D^L$ is the game

$$D^L = \{U,V,W,...\mid X,Y,Z,...\},$$

so that $G$ looks something like Fig. 6(a).

![Diagram](image)

---

Now where can move to any letting him move shown in Fig. 6. We can easily

shown in Fig. 7

$$\{A\}$$

shown in Fig. 7

![Diagram](image)

Obviously I negatives from $-H$ to $-X$ or Left's hope $G - X$, worse for Right's move if namely:

$$\{U\},$$

Now Right da from $-H$ to th from $-H$, leave are at least as the appropriate
REVERSIBLE MOVES IN GENERAL GAMES

Now whenever Right plays from \( G \) to \( D \), Left will reverse from \( D \) to \( D' \), from which Right can move to any of \( X, Y, Z, ... \). So we might as well shorten \( G \) by omitting Right's move to \( D \) and letting him move directly to \( X \) or \( Y \) or \( Z \) or ... . In this way, we get the game
\[
H = \{A,B,C,...|X,Y,Z,...,E,F,...\},
\]
shown in Fig. 6(b), which should have the same value as \( G \).

We can easily test this by playing the game \( G - H \), that is
\[
\{A,B,C,...|D,E,F,...\} + \{-X,-Y,-Z,...,-E,-F,...\} - A - B - C,...\},
\]
shown in Fig. 7, and verifying that there is no good move for either player as follows.

\[\text{Figure 7. A Zero Game.}\]

Obviously the moves from \( G \) to \( A,B,C,... \) or \( E,F,... \) are exactly countered by moves to their negatives from \( -H \), and conversely, so that the only hopeful moves are those for Left from \( -H \) to \( -X \) or \( -Y \) or \( -Z \) or ..., and that for Right from \( G \) to \( D \).

Left's hopes are soon dashed. His move from \( -H \) to \( -X \), say, leaves the total position \( G - X \), worse for him than \( D' - X \), which Right can win by moving from \( D' \) to \( X \). There remains Right's move from \( G \) to \( D \), which Left will reverse to \( D' \), leaving the total position \( D' - H \), namely:
\[
\{U,V,W,...|X,Y,Z,...\} + \{-X,-Y,-Z,...,-E,-F,...\} - A - B - C,...\}.
\]

Now Right dare not move from \( D' \) to \( X \) or \( Y \) or \( Z \) or ... , since Left can counter by moving from \( -H \) to the corresponding one of \( -X \) or \( -Y \) or \( -Z \) or ... . So Right's only hope is to move from \( -H \), leaving the total position \( D' - A \) or \( D' - B \) or \( D' - C \) or ... . But since \( D' \geq G \) these are at least as bad for Right as \( G - A \), \( G - B \), \( G - C \), ..., which Left can win by moving in \( G \) to the appropriate one of \( A \) or \( B \) or \( C \) or ... .
Since we have now dealt with all possible first moves, \( G - H \) is a zero game, and we can afford to replace \( G \) by \( H \) in any of our calculations, which will often be a very valuable simplification. We summarize:

If any Right option \( D \) of \( G \) has itself a Left option \( D^L \geq G \), then it will not affect the value of \( G \) if we replace \( D \) as a Right option of \( G \) by all the Right options \( X, Y, Z, \ldots \) of that \( D^L \).

**BYPASSING RIGHT'S REVERSIBLE MOVE**

Of course a move by Left can also be reversible:

If any Left option \( C \) of \( G \) has itself a Right option \( C^R \leq G \), then it will not affect the value of \( G \) if we replace \( C \) as a Left option by the list of all Left options of that \( C^R \).

**BYPASSING LEFT'S REVERSIBLE MOVE**

**DELETING DOMINATED OPTIONS**

Now there is another kind of simplification we've already mentioned, which it would be wise to discuss more precisely here. In the game

\[
G = \{A, B, C, \ldots | D, E, F, \ldots\},
\]

if \( A \leq B \) we say that \( A \) is dominated by \( B \), and if \( D \leq E \), that \( E \) is dominated by \( D \). In other words, given two possible moves for the same player, one dominates the other if it is at least as good for the person making it. Then we can simplify by omitting dominated moves (provided we retain the moves that dominate them). In the case discussed, this will mean that \( G \) has the same value as the game

\[
K = \{B, C, \ldots | D, F, \ldots\}.
\]

And indeed, \( G - K \) is a zero game, since the moves from \( G \) to \( A \) or \( E \) are countered by those from \(-K\) to \(-B\) or \(-D\), and all other moves in either component are countered by moves to their negatives from the other.

It won't affect the value of \( G \) if we delete dominated options but retain the options that dominated them.

**DELETING DOMINATED OPTIONS**
But remember that *reversible* options are *not* deleted, but *bypassed*, i.e. replaced by the list of options, for the appropriate player, from the position his opponent reverses to.

**TOADS-AND-FROGS WITH UPS AND DOWNS**

![Diagram of Toads-and-Frogs]

*Figure 8. Anatomy of Toads-and-Frogs.*
We considered a 4-place version of this game in Chapter 1. The 5-place version we now consider displays more interesting behaviour. In any 5-place lane Left may move one of his toads one space right, onto an empty square, or jump over just one frog onto an empty square immediately beyond. Right’s moves are similar, moving his frogs leftwards. Figure 8 shows the complete play from the initial position in which two toads are separated by just one space from two frogs. We already know how to evaluate, working upwards, all positions except the top three.

The next position to be considered is

\[ T \square T F F = \{ \square T F F \} = \{ \emptyset \} \cdot \]

Since \( \emptyset \) is a loss for the player to move from it, Left can win this game by moving to \( 0 \), and Right’s move to \( \ast \) does not win since Left will reply to \( 0 \). So \( \{ \emptyset \} \) is a positive game. But since \( \ast \) is less than each of the numbers

\[ 2, 1, \frac{1}{2}, \ldots \]

\( \{ \emptyset \} \) is less than or equal to each of

\[ \{ \emptyset \}, \{ \emptyset \}, \{ \emptyset \}, \{ \emptyset \}, \ldots \]

that is, each of

\[ 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \]

So we have here a positive value less than every positive number. Since we have not seen such a thing before, we cannot hope to simplify it, and we therefore need a new name, \( \uparrow \), pronounced “up”.

Similarly, the position \( T F F \square \), obtained by interchanging the roles of Left and Right, has the value \( \{ \ast \} \), which is negative, but greater than every negative number. Since \( \ast \) is its own negative, \( \{ \ast \} \) is the negative of \( \{ \emptyset \} \) and we call it \( \downarrow \), pronounced “down”. By the end of the book we shall have had many ups and downs!

So the starting position of Fig. 8 has value

\[ T \square T F F = \{ T \square T F F \} \cdot \]

Do we need a new name for this? Let’s first see if we can simplify it. Since each player has only one option, no move dominates another, so we next look for reversible moves. Is Right’s move to \( \uparrow \) reversible from the game \( \{ \uparrow \} \)? This happens only if there is some Left option \( \downarrow \geq \) \( \uparrow \). Since \( \downarrow = \{ \ast \} \), we are asking if \( \ast \geq \) \( \uparrow \), i.e. if \( \uparrow - \ast \leq 0 \). Has Left a winning move from

\[ \{ \uparrow \} = \{ \ast \} + \{ \emptyset \} \]

His move from \( \ast \) to \( 0 \) is parried by Right’s reply from \( \uparrow \) to \( \downarrow \), and his move from \( \uparrow \) to \( \uparrow \) is countered by Right’s response from \( \uparrow \) to \( \ast \), which leaves the total value \( \ast + \ast = 0 \). So indeed if Left starts Right wins, showing that \( \uparrow - \ast \leq 0 \). This means that Right can bypass his move to \( \downarrow \) by moving directly from \( \uparrow \) to \( \dot{\uparrow} \) \( = \ast \), and shows that \( \uparrow = \{ \ast \} \). In this Left can also bypass his move, so that \( \uparrow \) simplifies further to \( \{ \emptyset \} = \ast \).

\[ \{ \uparrow \} = \{ \ast \} = \{ \emptyset \} = \{ \emptyset \} = \ast \]

We could otherwise have seen that \( \uparrow = \ast \) by observing that Right also had no winning move from \( \uparrow - \ast \). However, for a more complicated \( \uparrow \) we can ask which moves are reversible even before we have guessed the simplest form of \( \uparrow \).
GAME TRACKING AND IDENTIFICATION

Much of this book is about finding out who wins various partizan games from arbitrary positions. Partizan games, we recall from the Extras to Chapter 1, are those in which the options available to the two players are not necessarily the same. For example, to find the winner in the 9-lane 5-place Toads-and-Frogs position of Fig. 9, we must work with sums of terms whose values may be $\uparrow$, $\downarrow$, $\ast$, or various numbers. When stalking other game we shall need to know how to add even more general values and find when the result is positive, negative, zero or fuzzy. We learned how to add numbers at school and we can add any small enough numbers using our nim-addition table (Table 2). We also know that $x+\ast = \{x|x\} = \ast x$ for any number $x$. So perhaps the simplest pair of values we have not yet added are $\uparrow$ and $\ast$. We shall call their sum $\uparrow\ast$. Is it perhaps equal to $\{\uparrow\ast\}$?

![Diagram of Toads and Frogs game](image)

**Figure 9.** Toads and Frogs make easy Big Game.

We can always test for equality of two values by seeing if their difference is a win for the second player. Is this true of the difference game

$$\{\uparrow\ast\} - \uparrow\ast = \{\uparrow\ast\} + \{+0\} + \{0\},$$

do you think? No! If Left makes his move to $\ast$ from the component $\uparrow$, the total value becomes $\{\uparrow\ast\} = \{\uparrow\ast\} + \{\ast\} + \{0\}$ which is clearly positive, since Left can win and Right can't. In fact it turns out that Right has no good move from $\{\uparrow\ast\} - \uparrow\ast$ so that $\{\uparrow\ast\}$ is strictly greater than $\uparrow\ast$. 

We shall find a correct formula for $\uparrow \ast$. From the definition of the sum of two games we have, by considering the moves of Left and Right in the two components,

$$\uparrow \ast = \uparrow \ast = \{0,\ast, \uparrow 0 | \ast, \ast \}$$

using the equation $\ast + \ast = 0$. We can simplify this to

$$\uparrow \ast = \{\ast \mid 0\}$$

since Right's option $\uparrow$ was dominated by his other option, 0. Neither of Left's options dominates the other, for from their difference $\uparrow - \ast = \uparrow \ast$, Left can win by moving to $\uparrow + 0$ and Right by moving to $\ast + \ast$.

However, Left's option $\uparrow$ is reversible by Right through $\uparrow^R = \ast$, for plainly $\ast < \uparrow$. So we can bypass $\uparrow$ by allowing Left to move directly to $\uparrow^R \ast = \ast$, without affecting the value of the game. We therefore have the equation

$$\uparrow \ast = \uparrow \ast = \{0, \ast | 0\}$$

and similarly its negative

$$\downarrow \ast = \downarrow \ast = \{0 | \ast\}.$$
WHAT ARE FLOWERS WORTH

(a)  (b)

Figure 10. Two Flower Shows Ready for Judging.

Here it is easy to see that whoever first chops one of the two green stalks will lose, for his opponent will chop the other. So this kind of game reduces to “She-Loves-Me, She-Loves-Me-Not”, this time played on the red and blue colored petals. Whichever of Left and Right is first unable to remove a petal of his color will lose, since his only other options are the stalks. So in a two-flower position the player having the larger number of petals of his color will win, except that if there are as many red as blue petals in all, the second player will win, for his opponent must take the first petal and hence the first stalk.

This argument proves that Fig. 10(b) is a zero game, for each player has three petals in all, and it establishes that the value of the flower of Fig. 5(a) in Chapter 2 is \( \downarrow + \ast \), since the one-petal flower in Fig. 10(b) has value \( \downarrow + \ast \) (the petal can be uncurled without affecting play, like a snake’s head).

A GALLIMAUFRY OF GAMES

Figure 11. A Gallimaufry of Games.
Left and Right will soon return to the table shown in Fig. 11, on which they have been playing the sum of three games, namely Hackenbush, Hotchpotch, Col, and Toads-and-Frogs. You can be sure that they will be unable to agree whose turn it was to move. Will it matter?

**WHO WINS SUMS OF UPS, DOWNS, STARS AND NUMBERS?**

A 5-place Toads-and-Frogs position with any number of lanes has a value which is the sum of terms from

0, 1, -1, \( \frac{1}{2} \), \( -\frac{1}{2} \), \( \ast \), \( \uparrow \) and \( \downarrow \)

To work out who wins we need rules telling us when such a sum is positive, negative, zero or fuzzy. Using the equations \( \ast + \ast = 0 \), and \( \uparrow = -1 \), any such sum reduces to a form \( x + n, \uparrow \) or \( x + n, \downarrow + \ast \), where \( x \) is a number and \( n \) an integer which may be positive, negative or zero. The rules (valid for arbitrary numbers \( x \)) are:

- **If \( x \) is any number, then \( x + n, \uparrow \) is**
  - positive, if \( x \) is positive, or \( x \) is zero and \( n \geq 1 \);
  - negative, if \( x \) is negative, or \( x \) is zero and \( n \leq -1 \);
  - and zero, only if \( x \) and \( n \) are both zero.

- **If \( x \) is any number, then \( x + n, \downarrow + \ast \) is**
  - positive, if \( x \) is positive, or \( x \) is zero and \( n \geq 2 \);
  - negative, if \( x \) is negative, or \( x \) is zero and \( n \leq -2 \);
  - and fuzzy, if \( x \) is zero and \( n = -1, 0 \) or 1.

In these rules, \( \uparrow, \downarrow \) denotes the sum of \( n \) copies of \( \uparrow \), or \( -n \) copies of \( \downarrow \). We usually abbreviate \( 2, \uparrow \) to \( \uparrow \), and write \( \uparrow \ast \) for \( \uparrow + \ast \), etc.

The proofs require only the observations we have already made that \( \ast \) is fuzzy and \( \uparrow \) positive and that both are dominated by any positive number, together with the observations that \( \uparrow \ast \) is fuzzy but \( \uparrow \ast \uparrow \) is positive.

To see that \( \uparrow \ast \) is fuzzy we need only observe that from the equivalent form \( \{0, \ast\}0 \) each player has a (winning) move to 0. From

\[
\uparrow \ast = \{0\ast\}+\{0\ast\}+\{0\}.
\]

Right's only options are to replace a component \( \uparrow \) by \( \ast \), leaving a total of \( \uparrow + \ast + \ast = \uparrow \), or to replace \( \ast \) by 0, leaving \( \uparrow \). Since these are both positive, Right has no winning option, so that \( \uparrow \ast \geq 0 \). Since \( \uparrow \) is positive it cannot equal \( \ast \), so \( \uparrow \ast \) must be strictly positive. In fact Left wins by replacing \( \ast \) by 0, leaving the positive remainder \( \uparrow \).
A CLOSER LOOK AT THE STARS

We have now acquired a better idea of how fuzzy * really is, for we have shown that it is less than \( \uparrow = \uparrow + \uparrow \), greater than \( \downarrow = \downarrow + \downarrow \), but confused with each of \( \downarrow, 0, \uparrow \). The cloud under which it is hiding, see Fig. 12, although it covers only one number, 0, can now be seen to have a radius of at least \( \uparrow \).

Figure 12. Star. Seen Through a Glass, Darkly.

We can examine other small games using similar devices. Fig. 13(a) shows \( \uparrow * \), obtained by adding \( \uparrow \) to Fig. 12. Figure 13(b) will serve for any \( *n \) with \( n \geq 2 \).

Figure 13. The Whereabouts of \( 1^* \) and of \( *n \ (n \geq 2) \).

THE VALUES \{\uparrow|\uparrow\} AND \{0|\uparrow\}

In more complicated positions, \( \uparrow \) and \( \downarrow \) frequently arise as options. For example, we have already seen that \( \{\uparrow|\downarrow\} = \{\uparrow|0\} = \{0|\uparrow\} = \ast \), and enquired about the position \{\ast|\ast\}.

The value \( \{0|\uparrow\} \) arises from the 7-place Toads-and-Frogs position of Fig. 14, in which the four positions marked 0 may be checked to be second player wins. How big are \{\uparrow|\uparrow\} and \{0|\uparrow\}?
We first examine \( \{1\} \) = \( X \), say. Right’s option of \( \uparrow \) will only be reversible if there is some \( \uparrow^L \geq X \), i.e. if \( 0 \geq X \), which we know is false. As a Left option, \( \uparrow \) will be reversible if there is some \( \uparrow^R \leq X \), i.e. if \( * \leq X \), which is true since

\[
* = \{0|0\} \leq \{1|1\} = X.
\]

So we can bypass, replacing \( \uparrow \) by \( *^L \) = 0, to obtain \( X = \{0|\} \), the value of Fig. 14, proving that our two questions were the same. Since 0 has no right option there will be no further simplification.

Is \( 0 \geq X \)? No! So no simplification.

\[
\begin{align*}
\text{Is } * \leq X? & \quad \text{Yes!} \\
\text{So } X & = \{0 | \uparrow \}
\end{align*}
\]

Figure 15. Searching for Reversible Moves.
THE UPSTART EQUALITY

For a general X, each player must ask if to any one of his opponent's options from X he has a response Y which is at least as good for him as X was. If so, he replaces that option by the list of all his opponent's options from Y. Figure 15 shows a graphical way of asking these questions that we have often found useful. The arrows are curved so as to remind us which players make which of the moves we hope to bypass.

THE UPSTART EQUALITY

How big is $X = \{9 \uparrow \downarrow \}$ on our microscopic scale? It is certainly less than $4 \uparrow$ (the sum of 4 copies of $\uparrow$), since in the difference $X + 4 \downarrow$, Right can move from $X$ to $\uparrow$ at his first opportunity and there will be at least two $\downarrow$ components, even after cancelling $\uparrow$ with $\downarrow$. By a similar move Right can win $X + 3 \downarrow$ if he moves first. However, Left can also win this moving first if he replaces $\downarrow$ by $*$, leaving

$$X + \downarrow * = \{9 \uparrow \downarrow \} + \{\downarrow 0 \} + \{\downarrow 0 \} + \{0 0 \}.$$

To see this, recall that we already know that $X$, alias $\{9 \uparrow \downarrow \}$, is strictly greater than $\uparrow *$, so Left wins if Right replaces a $\downarrow$ by $0$, while if Right replaces $*$ by $0$, Left can win by replacing a $\downarrow$ by $*$. Right's only other option is from $X$ to $\uparrow$ leaving a fuzzy total of $\downarrow *$.

The argument has shown that $X$ is confused with $3 \uparrow$, and so with $\uparrow$ and $\downarrow$, since even from $X + \downarrow$ Left has a winning move (to $X + *$). We now know all order relations between $X$ and values of the form $n \uparrow$. How does it compare with values $n, \uparrow + + *$? Since it is greater than $\uparrow *$ we compare it with $\downarrow *$. In the difference $X + \downarrow *$, displayed above, we have already dismissed all Right's options. However, Left's option from $X$ leaves the negative total $\downarrow *$; his option from $\downarrow$ leaves the fuzzy total $X + + + + * = X + \downarrow$, while that from $*$ leaves another fuzzy total $X + \downarrow$, so all Left's options can be dismissed too! This gives us the remarkable identity

$$\{9 \uparrow \downarrow \} = \uparrow + \uparrow + * = \downarrow *.$$

The theory of partizan games is notable for the occurrence of such surprising identities. Although the pattern in Table 3 extends naturally in both directions, some of the middle entries are far

| $3, \downarrow = \{\downarrow 0 \}$ | $3, \downarrow + * = \{\downarrow 0 \}$ | $3, \downarrow + n = \{\downarrow + n \}$ |
| $\downarrow = \{\downarrow 0 \}$ | $\downarrow * = \{\downarrow 0 \}$ | $\downarrow + n = \{\downarrow + n \}$ |
| $\downarrow = \{\downarrow 0 \}$ | $\downarrow + = \{\downarrow + = \} \}$ | $\downarrow + = \{\downarrow + = \}$ |
| $0 = \{ \}$ | $\downarrow + = \{\downarrow 0 \}$ | $\downarrow = \{\downarrow 0 \}$ |
| $1 = \{ \}$ | $\downarrow + = \{\downarrow 0 \}$ | $\downarrow = \{\downarrow 0 \}$ |
| $1 = \{\downarrow 0 \}$ | $\downarrow + = \{\downarrow 0 \}$ | $\downarrow = \{\downarrow 0 \}$ |
| $\downarrow = \{\downarrow 0 \}$ | $\downarrow + = \{\downarrow 0 \}$ | $\downarrow = \{\downarrow 0 \}$ |
| $\downarrow = \{\downarrow 0 \}$ | $\downarrow + = \{\downarrow 0 \}$ | $\downarrow = \{\downarrow 0 \}$ |

Table 3. Simplest Forms for Ups and Stars.
from immediately obvious. In the last column \(\ast\) denotes the number \(\{0,\ast,\ast(n-1)\}0,\ast,\ast(n-1)\}\) for some \(n \geq 2\) and \(m = n \uparrow\downarrow\); \(\ast n\) denotes \(\uparrow\downarrow \ast n\), etc.

In particular, these relationships allow us to obtain a tractable expression for Toads-and-Frogs positions of the form

\[(TF)^kT \square (TF)^oF.\]

Omar will already notice that the Toad move gives a position of value 0; our less assiduous readers will find that this is a consequence of a more general result in Chapter 5. It follows that

\[(TF)^kT \square (TF)^oF = \{0|TF\}^s \uparrow T \square (TF)^oF^o\]

which equals \(n \uparrow (n+1)\ast\) by induction on \(n\) (and doesn’t depend on \(s\)).

**GIFT HORSES**

The following principle often makes it easy to check one’s guess about the value of a position:

> It does not affect the value of \(G\)

if we add a new Left option \(H\) provided \(H \leftarrow G\),
or a new Right option \(\overline{H}\) provided \(\overline{H} \leftarrow G\).

**THE GIFT HORSE PRINCIPLE**

The new options \(H\) or \(\overline{H}\) are the gift horses; although they may appear to be useful presents, the recipient who looks them in the mouth will find that they have no teeth. For in the difference game

\[
\{G^L, H|G^R\} + \{-G^R| -G^L\},
\]

if \(H\) is a gift horse, Left will find no joy in moving to the difference \(H - G \leftarrow 0\), and the other options for Left and Right cancel each other as in the Tweedledum and Tweedledee Argument.

Thus, we know that \(\{0|1\} = \ast \uparrow\downarrow 1\) is confused with \(\uparrow\downarrow\), so \(\uparrow\downarrow\) will make a fine gift horse for Left;

\(\{0|\} = \{0,1|\}\). Since Left’s old option 0 becomes dominated in the new form, we can deduce \(\{0|\} = \{1|\}\) more simply than we did before. In fact we have \(\ast \leftarrow 1\) and \(\uparrow\downarrow \leftarrow 3\), and so by a similar argument

\[
\uparrow \ast = \{0|1\} - \{1|\} - \{1|\} = \{3,\uparrow|1\}\]

On the other hand \(\uparrow \ast < 4,\uparrow\), so the latter would not be a mere gift horse for Left, and indeed \(\{4,1|\}\) is strictly greater than \(\uparrow \ast\).
EXTRAS

THE NIM-ADDITION RULE IN SEVERAL VARIATIONS

If you think about the way the nim-addition table (Table 2) extends, you'll see that

if $a$ and $b$ are less than $2^k$,
then so is $a \oplus b$, and
$2^k \oplus a = 2^k \oplus a$.

From this you can deduce that

the nim-sum of a number of different powers of 2
is their ordinary sum, and, of course,
the nim-sum of two equal numbers is zero.

THE BASICS OF NIM-ADDITION

You can use these two basic properties to find the nim-sum of any collection of numbers by writing each of them as a sum of distinct powers of 2 and then cancelling repetitions in pairs. For example,

$$5 \oplus 3 = (4+1) \oplus (2+1) = 4 \oplus 2 \oplus 1 = 4 = 2 - 4 + 2 - 4 = 6,$$

$$11 \oplus 22 \oplus 33 = (8+3+2) \oplus (16+4+2) \oplus (32+1) = 8 + 16 + 4 + 32 = 60.$$ 

These could also be written directly in terms of numbers,

4 + 3 = 6 and 11 + 22 + 33 = 60,

and you should get used to working in either notation:

$$9 \oplus 25 \oplus 49 = (8 \oplus 1) \oplus (16 \oplus 8 \oplus 1) \oplus (32 \oplus 16 \oplus 1) = 32 \oplus 1 = 33.$$ 

This way of calculating shows you that

the nim-sum is less than or equal to the ordinary sum,
and they differ by an even number.

SEE HOW THE SUMS COMPARE AND HAVE COMMON PARITY

The textbooks usually say "write the numbers in binary and add without carrying" which comes to the same thing:
SOME HARDER GAMES AND HOW TO MAKE THEM EASIER

\[
\begin{array}{cccc}
4 & 2 & 1 \\
32 & 16 & 8 & 4 & 2 & 1 \\
32 & 16 & 8 & 4 & 2 & 1 \\
\hline
5 & 1 & 0 & 1 \\
22 & 1 & 0 & 1 & 1 \\
25 & 1 & 1 & 0 & 0 & 1 \\
6 & 1 & 1 & 0 \\
60 & 1 & 1 & 1 & 0 & 0 \\
33 & 1 & 0 & 0 & 0 & 1 \\
\end{array}
\]

But since you don’t want to be scribbling on bits of paper, you should use our way, which makes it easy to do the sums in your head, and is less prone to error.

WYT QUEENS AND WYTHOFF’S GAME

Table 4 gives the nimbers for various possible positions of Wyt Queens.

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<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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<th>10</th>
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<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
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<td>+16</td>
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<td>+19</td>
<td>+20</td>
<td>+21</td>
<td>+4</td>
</tr>
</tbody>
</table>

Table 4: Nimbers for Wyt Queens.

Most of the entries are chaotic, but Wythoff’s **Difference Rule** is that the zero entries have coordinates

(0,0), (1,2), (3,3), (4,7), (6,10), (8,13), (9,15), (11,18), ...

with differences

0, 1, 2, 3, 4, 5, 6, 7, ...

the first number in every pair being the smallest number that hasn’t yet appeared. He also showed that the nth pair is
TOAD VERSUS FROG

where \( \tau \) is the golden number

\[
\frac{1 + \sqrt{5}}{2}.
\]

ANSWERS TO FIGS. 8, 9 AND 11.

Now that we know that \( \{0|+\} = \uparrow, \{+|0\} = \downarrow \) and \( \{1|1\} = \ast \), we can fill in the values of the positions in the first two rows of Fig. 8. Then in the easy big game of Toads-and-Frogs shown in Fig. 9, the values of the 9 lanes are \( \ast, 0, 1, \frac{1}{2}, \ast, -1, \ast, \frac{1}{2} \) and \( \uparrow \), whose total is \( \uparrow \ast \), a win for Toads. But if Left has to play first, he must be careful and move in one of the star lanes, either in the starting position of lanes 1 or 7 to make the value \( -1 \), or in the middle lane, making the value \( \downarrow \).

In our Gallimaufry of Games (Fig. 11) the Hackenbush position has the value \( \uparrow \ast \); the Col position \( 1 \ast \); and the Toads-and-Frogs position \( -1 \). In their sum, \( 1 \) and \( -1 \) cancel, as do the two stars, leaving simply \( \uparrow \). This is a win for Left, no matter who starts.

TOAD VERSUS FROG

A special case of Toads-and-Frogs which we can analyze completely is when each lane contains just one toad and one frog. After some moves we might find that the toad confronts the frog, so that either could jump over the other into an empty space just beyond. We then have

\[
\begin{array}{c|c|c}
\text{a} + 1 \text{ spaces} & \text{toad} & \text{b} + 1 \text{ spaces} \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c}
\text{a} + 2 \text{ spaces} & \text{toad} & \text{b} \text{ spaces} & \text{frog} & \text{b} + 2 \text{ spaces} \\
\end{array}
\]

\[
= |b - a - 2| |b - a + 2|
\]

since after either jump is made we can see exactly how many moves each creature has left to make. So for such positions the value is \( \{d - 2 | d + 2\} \) where \( d \) is the difference

\[
(\text{number of spaces to right of frog}) - (\text{number of spaces to left of toad}).
\]

This rule also works if either of these parentheses is 0. In the general position before confrontation there will be \( c \) spaces between the two creatures and a move by either player will shorten this gap to \( c - 1 \) and decrease \( d \) by 1, if a toad move, increase it by 1 if a frog move. So in Table 5 the Left and Right options for each entry are the entries left and right of it in the row above, for \( c = 1, 2, 3, \ldots \) while for \( c = 0 \) they are \( d - 2 \) and \( d + 2 \).

Using this rule to compute the entries it is easy to see that the rows continue to alternate. In the position of Fig. 16 the values of the lanes are, in order, \( 1, \ast, -\frac{1}{2}, 0 \) and \( -\frac{1}{2} \), which add to \( \ast \), so that either player can win by moving in the second lane. Can the reader find Left's only other winning move?
TWO THEOREMS ON SIMPLIFYING GAMES

We prove that by omitting all dominated options and bypassing reversible ones we really do obtain the absolutely simplest form of any game with finitely many positions. For suppose that $G$ and $H$ are games which have the same value, but that neither of them has any position with dominated or reversible options. Then we shall prove that for every option of $G$ there is an equal option of $H$, for the same player, and vice versa, so that $G$ and $H$ are not only equal in value, but identical in form.

Since the difference game

$$G - H = \{G^L | G^R\} + \{-H^R | -H^L\}$$

is a second player win, Right must have some winning response

$$G^{L^2} - H \leq 0 \quad \text{or} \quad G^R - H^L \leq 0$$

to any one of Left’s options $G^L - H$. The first case would imply $G^{L^2} \leq H = G$, making $G^L$ a reversible move from $G$, so that for every $G^{L^2}$ there must be some $H^{L^2} \geq G^{L^2}$. By a similar argument there must be some $G^{L^2} \geq H^{L^2} (\geq G^L)$ and since there are no dominated options, in fact $G^{L^2} = H^{L^2} = G^{L^2}$. The argument works equally well if we interchange $G$ with $H$ or Left with Right.
BERLEKAMP'S RULE FOR HACKENBUSH STRINGS

Our second theorem is that from the simplest form one can obtain any other form by adding gift horses and then perhaps deleting some dominated options. For if \( G = \{ G^L \mid G^R \} \) is the simplest form of some game \( H \equiv \{ H^L \mid H^R \} \), we can prove as before that Right's winning move from \( G^L - H \) must be to some game \( G^L - H^L \leq 0 \) (rather than some \( G^{2R} - H \leq 0 \)).

This proves that for each \( G^L \) there is some \( H^* \geq G^L \), and similarly for each \( G^R \), some \( H^R \leq G^R \). Also for every \( H^L \) we must have \( H^L \prec H \) and for every \( H^R, H^R \succ H \), since neither player can have a winning move from \( H - H = 0 \). The last sentence shows that the options of \( H \) will serve as gift horses for \( G \), so that

\[
G = \{ G^L, H^L \mid G^R, H^R \}
\]

by the Gift Horse Principle. In this form each option \( G^L \) or \( G^R \) is dominated by some \( H^L \) or \( H^R \) and so may be omitted.

BERLEKAMP'S RULE FOR HACKENBUSH STRINGS

Here's how to find a Hackenbush string for a given number. The color of the edge which touches the ground is taken from the sign of the number, so that positive numbers start with a blue edge and negative with a red one. We'll just do the positive case.

Write the fractional part of the number in binary; thus

\[
\frac{35}{8} = 3 \cdot 101.
\]

Then, to find the Hackenbush string, replace the integer part by a string of \( L \)'s, the point by \( LR \)

and convert 1's and 0's after the point into \( L \)'s and \( R \)'s, but **omitting the final digit**, 1:

\[
\frac{35}{8} = 3 \cdot 101 \\
L \ L \ L \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ L \ R \ \ L \ R
\]

Of course,

\[
-\frac{35}{8} = R \ R \ R \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ R \ L \ \ R \ L.
\]

The rule actually works even for real numbers which don't terminate, except that there is no final 1 to be omitted. E.g.,

\[
\frac{1}{7} = 0 \cdot 01010101 \ldots \\
L \ R \ R \ L \ R \ L \ R \ L \ R \ L \ R \ L \ R \ L \ R \ L \ R \ L \ R \ L \ R \ L \ R \ L \ R \ L \ R \ L \ R \ L \ R \ L \ R \ L \ R \ L \ R \ L \ R \ L \ R \ L
\]

Of course, the rule can be reversed to convert any Blue-Red Hackenbush string to a number. For example:

\[
0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1
\]

becoming \( \frac{7}{8} \).
SOME HARDER GAMES AND HOW TO MAKE THEM EASIER

We write this as a string of L's and R's and replace the first pair of adjacent branches of different colors by a point, convert subsequent branches by the rule:

- a color agreeing with the grounded color becomes 1,
- a color opposite to the grounded color becomes 0,

and append an extra 1 bit at the end. Thus

\[ R R R R \ R \ L \ R R \ L \ L \]

becomes

\[ -4 \ \\
0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 1 \]

i.e.

\[ -4 + \frac{1}{8} + \frac{1}{32} + \frac{1}{128} = -4\frac{63}{128}. \]

In certain applications when one wishes to store numbers whose distribution is known a priori, this Hackenbush number system may have significant advantages over the more conventional computer representations of fixed or floating point numbers.

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