Appendix 1: Utility Theory

Much of the theory presented is based on utility theory at a fundamental level. This theory gives a justification for our assumptions (1) that the payoff functions are numerical valued and (2) that a randomized payoff may be replaced by its expectation. There are many expostions on this subject at various levels of sophistication. The basic theory was developed in the book of Von Neumann and Morgenstern (1947). Further developments are given in Savage (1954), Blackwell and Girshick (1954) and Luce and Raiffa (1957). More recent expositions may be found in Owen (1982), Shubik (1984), Straffin (1993). Here is a brief description of the basics of utility theory.

The method a 'rational' person uses in choosing between two alternative actions, a_1 and a_2, is quite complex. In general situations, the payoff for choosing an action is not necessarily numerical, but may instead represent complex entities such as "you receive a ticket to a ball game tomorrow when there is a good chance of rain and your raincoat is torn" or "you lose five dollars on a bet to someone you dislike and the chances are that he is going to rub it in". Such entities we refer to as *payoffs* or *prizes*. The 'rational' person in choosing between two actions evaluates the value of the various payoffs and balances it with the probabilities with which he thinks the payoffs will occur. He may do, and usually does, such an evaluation subconsciously. We give here a mathematical model by which such choices among actions are made. This model is based on the notion that a 'rational' person can express his preferences among payoffs in a method consistent with certain axioms. The basic conclusion is that the value to him of a payoff may be expressed as a numerical function, called a *utility*, defined on the set of payoffs, and that the preference between actions giving him a probability distribution over the payoffs is based only on the expected value of the utility of the action.

Let $\mathcal{P}$ denote the set of payoffs of the game. We use P, P_1, P_2, and so on to denote payoffs (that is, elements of $\mathcal{P}$).

Definition. *A* **preference relation on** $\mathcal{P}$, *or simply preference on* $\mathcal{P}$, *is a (weak) linear ordering,* $\preceq$, *on* $\mathcal{P}$*; that is,*

(a) (linearity) if P_1 *and* P_2 *are in* $\mathcal{P}$, *then either* $P_1 \preceq P_2$ *or* $P_2 \preceq P_1$ *(or both), and*

(b) (transitivity) if P_1, P_2 *and* P_3 *are in* $\mathcal{P}$, *and if* $P_1 \preceq P_2$ *and* $P_2 \preceq P_3$, *then* $P_1 \preceq P_3$.

If $P_1 \preceq P_2$ *and* $P_2 \preceq P_1$, *then we say* P_1 *and* P_2 *are* **equivalent** *and write* $P_1 \simeq P_2$.

We assume that our 'rational' being can express his preferences over the set $\mathcal{P}$ in a way that is consistent with some preference relation. The statement $P_1 \preceq P_2$ means that our rational person either prefers P_2 to P_1 or he is indifferent between them. If $P_1 \simeq P_2$, we say that he is **indifferent** between P_1 and P_2.

Unfortunately, just knowing that a person prefers P_2 to P_1, gives us no indication of how much more he prefers P_2 to P_1. In fact, the question does not make sense until a third point of comparison is introduced. We could, for example, ask him to compare P_2 with the joint payoff of P_1 and $100 in order to get some comparison of how much more he prefers P_2 to P_1 in terms of money. We would like to go farther and express all his preferences in some numerical form. To do this however requires that we ask him to express his preferences on the space of all lotteries over the payoffs.

Definition. *A* **lottery** *is a finite probability distribution over the set $\mathcal{P}$ of payoffs. We denote the set of lotteries by $\mathcal{P}^*$.*

(A finite probability distribution is one that gives positive probability to only a finite number of points.)

If P_1, P_2 and P_3 are payoffs, the probability distribution, p, that chooses P_1 with probability 1/2, P_2 with probability 1/4, and P_3 with probability 1/4 is a lottery. We use lower case letters, p, p_1, p_2 to denote elements of $\mathcal{P}^*$. Note that the lottery p that gives probability 1 to a fixed payoff P may be identified with P, since receiving payoff P is the same as receiving payoff P with probability 1. With this identification, we may consider $\mathcal{P}$ to be a subset of $\mathcal{P}^*$.

We note that if p_1 and p_2 are lotteries and $0 \le \lambda \le 0$, then $\lambda p_1 + (1 - \lambda)p_2$ is also a lottery. It is that lottery that first tosses a coin with probability λ of heads; if heads comes up, then it uses p_1 to choose an element of $\mathcal{P}$ and if tails comes up, it uses p_2. Thus $\lambda p_1 + (1 - \lambda)p_2$ is an element of $\mathcal{P}^*$. Mathematically, a lottery of lotteries is just another lottery.

We assume now that our 'rational' person has a preference relation not only over $\mathcal{P}$ but over $\mathcal{P}^*$ as well. One very simple way of setting up a preference over $\mathcal{P}^*$ is through a utility function.

Definition. *A* **utility function** *is a real-valued function defined over $\mathcal{P}$.*

Given a utility function, $u(P)$, we may extend the domain of u to the set $\mathcal{P}^*$ of all lotteries by defining $u(p)$ for $p \in \mathcal{P}^*$ to be the expected utility: i.e. if $p \in \mathcal{P}^*$ is the lottery that chooses $P_1, P_2, \ldots, P_k$ with respective probabilities $\lambda_1, \lambda_2, \ldots, \lambda_k$, where $\lambda_i \ge 0$ and $\sum \lambda_i = 1$, then

$$u(p) = \sum_{i=1}^{k} \lambda_i u(P_i) \tag{1}$$

is the expected utility of the payoff for lottery p. Thus given a utility u, a simple preference over $\mathcal{P}^*$ is given by

$$p_1 \preceq p_2 \qquad \text{if and only if} \qquad u(p_1) \le u(p_2), \tag{2}$$

i.e. that lottery with the higher expected utility is preferred.

The basic question is, can we go the other way around? Given an arbitrary preference, $\preceq$ on $\mathcal{P}^*$, does there exist a utility u defined on $\mathcal{P}$ such that (2) holds? The answer is no in general, but under the following two axioms on the preference relation, the answer is yes!

A1. *If p_1, p_2 and q are in $\mathcal{P}^*$, and $0 < \lambda \le 1$, then*

$$p_1 \preceq p_2 \qquad \textit{if, and only if} \qquad \lambda p_1 + (1-\lambda)q \preceq \lambda p_2 + (1-\lambda)q. \tag{3}$$

A2. *For arbitrary p_1, p_2 and q in $\mathcal{P}^*$,*

$$\lambda q + (1-\lambda)p_1 \preceq p_2 \quad \textit{for all} \;\; 0 < \lambda < 1 \qquad \textit{implies} \qquad p_1 \preceq p_2 \tag{4}$$

and similarly,

$$p_2 \preceq \lambda q + (1-\lambda)p_1 \quad \textit{for all} \;\; 0 < \lambda < 1 \qquad \textit{implies} \qquad p_2 \preceq p_1. \tag{5}$$

Axiom A1 is easy to justify. Consider a coin with probability λ of coming up heads. If the coin comes up tails you receive q. If it comes up heads you are asked to choose between p_1 and p_2. If you prefer p_2, you would naturally choose p_2. This axiom states that if you had to decide between p_1 and p_2 before learning the outcome of the toss, you would make the same decision. A minor objection to this axiom is that we might be indifferent between $\lambda p_1 + (1-\lambda)q$ and $\lambda p_2 + (1-\lambda)q$ if λ is sufficiently small, say $\lambda = 10^{-100}$, even though we prefer p_1 to p_2. Another objection comes from the person who dislikes gambles with random payoffs. He might prefer a p_2 that gives him \$2 outright to a gamble, p_1, giving him \$1 with probability 1/2 and \$3.10 with probability 1/2. But if q is \$5 for sure and $\lambda = 1/2$, he might prefer $\lambda p_1 + (1-\lambda)q$ to $\lambda p_2 + (1-\lambda)q$ on the basis of larger expected monetary reward, because the payoff is random in either case.

Axiom A2 is more debatable. It is called the continuity axiom. It is safe to assume that for most people, \$100 is strictly preferred to \$1, which is strictly preferred to death. Yet, would you ever prefer a gamble giving you death with probability λ and \$100 with probability $1-\lambda$ for some positive λ to \$1 outright? If not, then with q = death, p_1 = \$100 and p_2 = \$1, condition (4) is violated. However, people do not behave as if avoiding death is an overriding concern. They will go on the freeway to get to work so they can earn some money, even though they have increased the probability of death (by a very small amount) by doing so. At any rate, Axiom A2 implies that there is no payoff infinitely less desirable or infinitely more desirable than any other payoff.

Theorem 1. *If a preference relation, $\preceq$, on $\mathcal{P}^*$ satisfies A1 and A2, then there exists a utility, u, defined on $\mathcal{P}$ that satisfies (2). Furthermore, u is uniquely determined up to change of location and scale.*

If a utility $u(P)$ satisfies (2), then for arbitrary real numbers a and $b > 0$, the utility $\hat{u}(P) = a + bu(P)$ also satisfies (2). Thus the uniqueness of u up to change of location and scale the strongest uniqueness that can be obtained.

Exercises. 1. Does every preference given by a utility as in (1) satisfy A1 and A2?

2. Take $\mathcal{P} = \{P_1, P_2\}$, and give an example of a preference on $\mathcal{P}^*$ satisfying A2 but not A1.

3. Take $\mathcal{P} = \{P_1, P_2, P_3\}$, and give an example of a preference on $\mathcal{P}^*$ satisfying A1 but not A2.

Appendix 2: Existence of Equilibria in Finite Games

We give a proof of Nash's Theorem based on the celebrated Fixed Point Theorem of L. E. J. Brouwer. Given a set C and a mapping T of C into itself, a point $\boldsymbol{z} \in C$ is said to be a fixed point of T, if $T(\boldsymbol{z}) = \boldsymbol{z}$.

Brouwer's Fixed Point Theorem. *Let C be a nonempty, compact, convex set in a finite dimensional Euclidean space, and let T be a continuous map of C into itself. Then there exists a point $\boldsymbol{z} \in C$ such that $T(\boldsymbol{z}) = \boldsymbol{z}$.*

The proof is not easy. You might look at the paper of K. Kuga (1974), "Brower's fixed point Theorem: An Alternate Proof", *SIAM Journal of Mathematical Analysis,* **5**, 893-897. Or you might also try Parthasarathy and Raghavan (1971), Chapter 1.

Now consider a finite n-person game with the notation of Section III.2.1. The pure strategy sets are denoted by $X_1, \ldots, X_n$, with X_k consisting of $m_k \geq 1$ elements, say $X_k = \{1, \ldots, m_k\}$. The space of mixed strategies of Player k is given by X_k^*,

$$X_k^* = \{\boldsymbol{p}_k = (p_{k,1}, \ldots, p_{k,m_k}) : p_{k,i} \geq 0 \text{ for } i = 1, \ldots, m_k, \text{ and } \textstyle\sum_{i=1}^{m_k} p_{k,i} = 1\}. \tag{1}$$

For a given joint pure strategy selection, $\boldsymbol{x} = (i_1, \ldots, i_n)$ with $i_j \in X_j$ for all j, the payoff, or utility, to Player k is denoted by $u_k((i_1, \ldots, i_n)$ for $k = 1, \ldots, n$. For a given joint mixed strategy selection, $(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n)$ with $\boldsymbol{p}_j \in X_j^*$ for $j = 1, \ldots, n$, the corresponding expected payoff to Player k is given by $g_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n)$,

$$g_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n) = \sum_{i_1=1}^{m_1} \cdots \sum_{i_n=1}^{m_n} p_{1,i_1} \cdots p_{n,i_n} u_k(i_1, \ldots, i_n). \tag{2}$$

Let us use the notation $g_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n | i)$ to denote the expected payoff to Player k if Player k changes strategy from p_k to the pure strategy $i \in X_k$,

$$g_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n | i) = g_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{k-1}, \boldsymbol{\delta}_i, \boldsymbol{p}_{k+1}, \ldots, \boldsymbol{p}_n). \tag{3}$$

where $\boldsymbol{\delta}_i$ represents the probability distribution giving probability 1 to the point i. Note that $g_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n)$ can be reconstructed from the $g_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n | i)$ by

$$g_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n) = \sum_{i=1}^{m_k} p_{k,i} g_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n | i) \tag{4}$$

A vector of mixed strategies, $(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n)$, is a strategic equilibrium if for all $k = 1, \ldots, n$, and all $i \in X_k$,

$$g_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n | i) \leq g_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n). \tag{5}$$

Theorem. *Every finite n-person game in strategic form has at least one strategic equilibrium.*

Proof. For each k, X_k^* is a compact convex subset of m_k dimensional Euclidean space, and so the product, $C = X_1^* \times \cdots \times X_n^*$, is a compact convex subset of a Euclidean space of dimension $\sum_{i=1}^n m_i$. For $\boldsymbol{z} = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n) \in C$, define the mapping $T(\boldsymbol{z})$ of C into C by

$$T(\boldsymbol{z}) = \boldsymbol{z}' = (\boldsymbol{p}_1', \ldots, \boldsymbol{p}_n') \tag{6}$$

where

$$p_{k,i}' = \frac{p_{k,i} + \max(0, g_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n|i) - g_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n))}{1 + \sum_{j=1}^{m_k} \max(0, g_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n|j) - g_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n))}. \tag{7}$$

Note that $p_{k,i} \geq 0$, and the denominator is chosen so that $\sum_{i=1}^{m_k} p_{k,i}' = 1$. Thus $\boldsymbol{z}' \in C$. Moreover the function $f(\boldsymbol{z})$ is continuous since each $g_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n)$ is continuous. Therefore, by the Brouwer Fixed Point Theorem, there is a point, $\boldsymbol{z}' = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n) \in C$ such that $T(\boldsymbol{z}') = \boldsymbol{z}'$. Thus from (7)

$$q_{k,i} = \frac{q_{k,i} + \max(0, g_k(\boldsymbol{z}'|i) - g_k(\boldsymbol{z}'))}{1 + \sum_{j=1}^{m_k} \max(0, g_k(\boldsymbol{z}'|j) - g_k(\boldsymbol{z}'))}. \tag{8}$$

for all $k = 1, \ldots, n$ and $i = 1, \ldots, m_n$. Since from (4) $g_k(\boldsymbol{z}')$ is an average of the numbers $g_k(\boldsymbol{z}'|i)$, we must have $g_k(\boldsymbol{z}'|i) \leq g_k(\boldsymbol{z}')$ for at least one i for which $q_{k,i} > 0$, so that $\max(0, g_k(\boldsymbol{z}'|i) - g_k(\boldsymbol{z}')) = 0$ for that i. But then (8) implies that $\sum_{j=1}^{m_k} \max(0, g_k(\boldsymbol{z}'|j) - g_k(\boldsymbol{z}')) = 0$, so that $g_k(\boldsymbol{z}'|i) \leq g_k(\boldsymbol{z}')$ for all k and i. From (5) this shows that $\boldsymbol{z}' = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n)$ is a strategic equilibrium. ■

Remark. From the definition of $T(\boldsymbol{z})$, we see that $\boldsymbol{z} = (p_1, \ldots, p_n)$ is a strategic equilibrium if and only if $\boldsymbol{z}$ is a fixed point of T. In other words, the set of strategic equilibria is given by $\{\boldsymbol{z} : T(\boldsymbol{z}) = \boldsymbol{z}\}$. If we could solve the equation $T(\boldsymbol{z}) = \boldsymbol{z}$ we could find the equilibria. Unfortunately, the equation is not easily solved. The method of iteration does not ordinarily work because T is not a contraction map.