Informed search methods

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Read Section 3.5-3.7 of Russell and Norvig

Informed Search Methods

Heuristic = "to find", "to discover"

- "Heuristic" has many meanings in general
  - How to come up with mathematical proofs
  - Opposite of algorithmic
  - Rules of thumb in expert systems
  - Improve average case performance, e.g. in CSPs
  - Algorithms that use low-order polynomial time (and come within a bound of the optimal solution)
    - % from optimum
    - % of cases
    - % PAC
  - h(n) that estimates the remaining cost from a state to a solution

Best-First Search

f(n)

function BEST-FIRST-SEARCH (*problem,* EVAL-FN) **returns** a solution sequence
  inputs: *problem*, a problem
    Eval-Fn, an evaluation function

  Queuing-Fn – a function that orders nodes by EVAL-FN
  return GENERAL-SEARCH (*problem, Queuing-Fn)*

An implementation of best-first search using the general search algorithm.

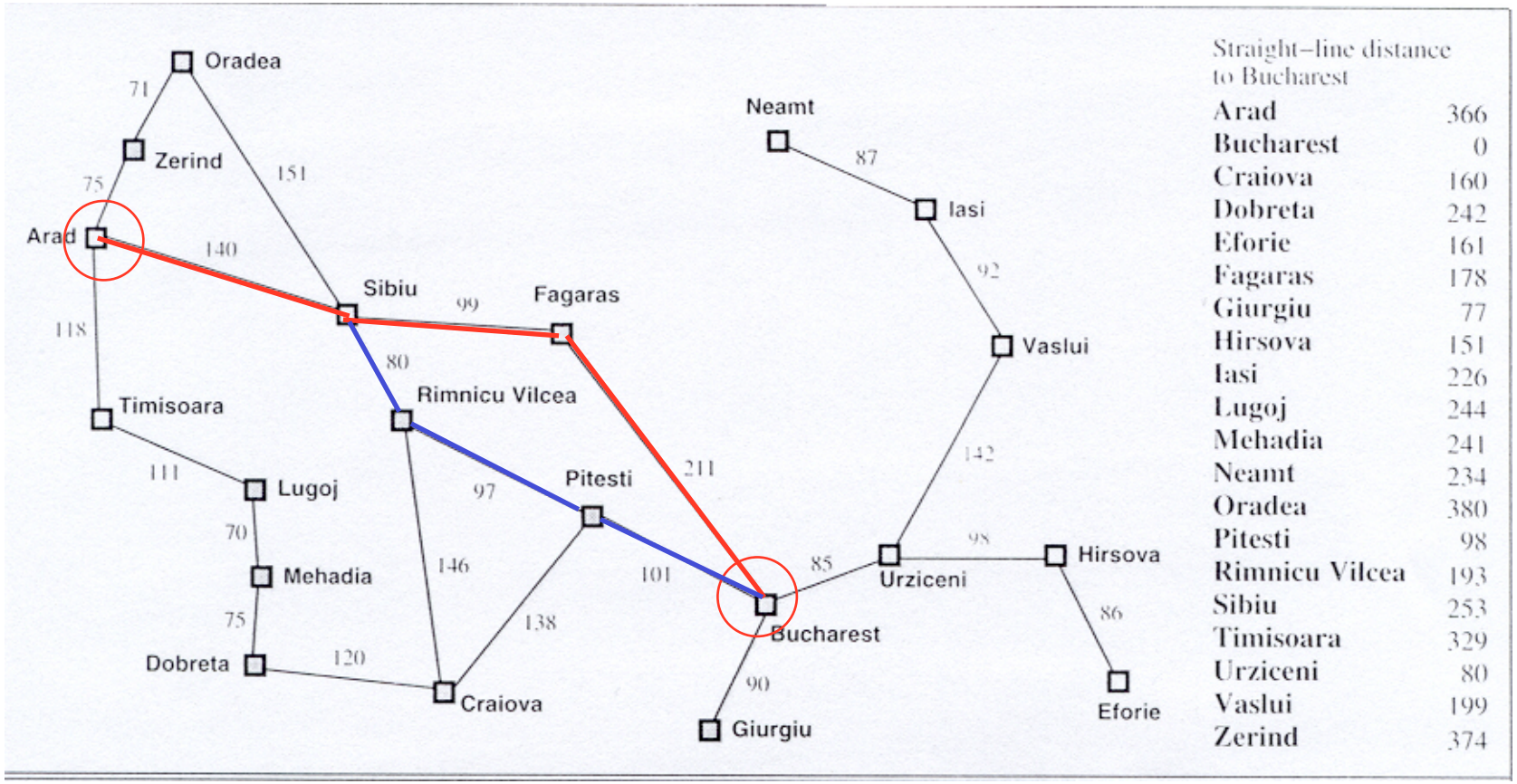
Usually, knowledge of the problem is incorporated in an evaluation function that describes the desirability of expanding the particular node.

If we really knew the desirability, it would not be a search at all.
"Seemingly best-first search"

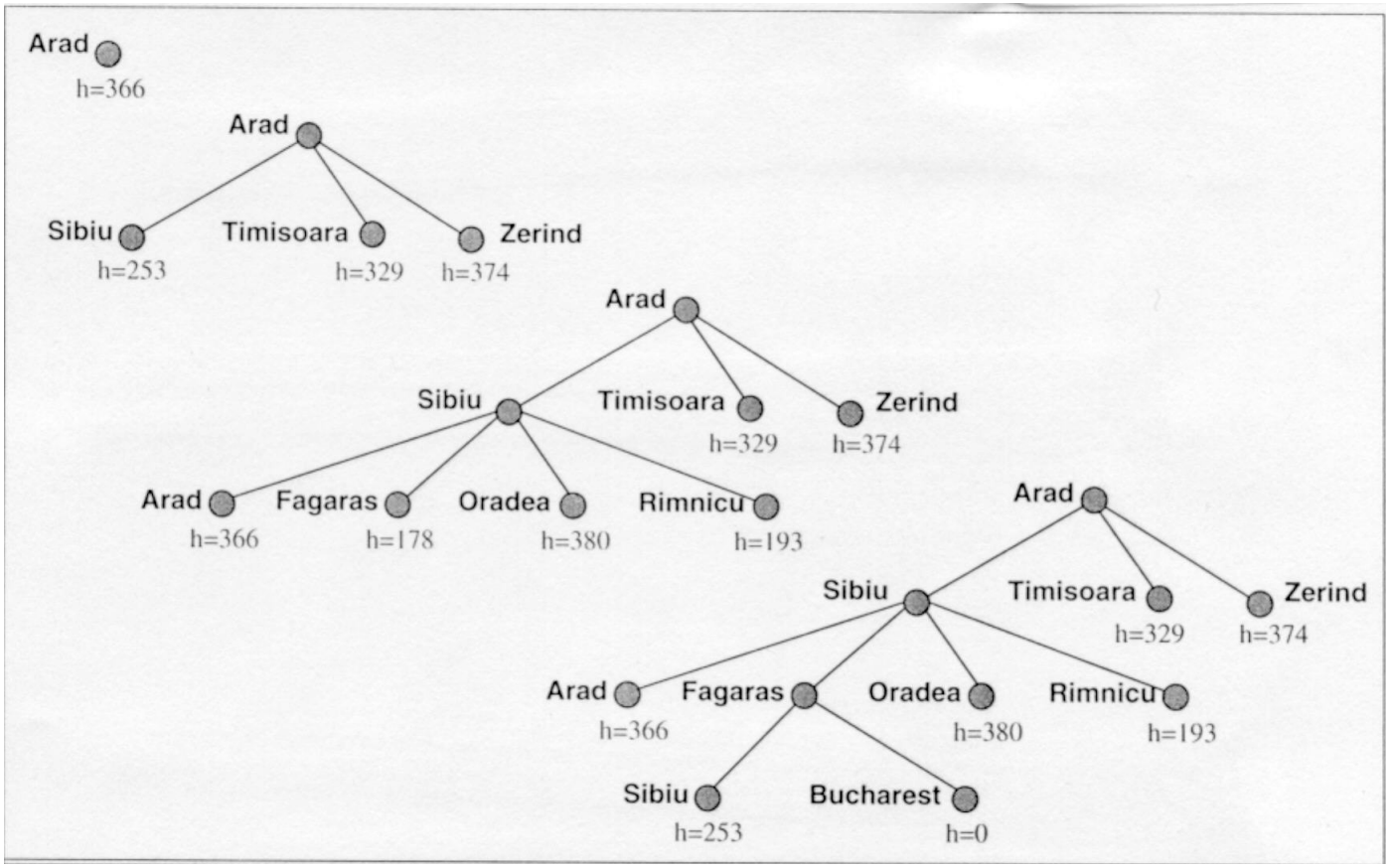
Greedy Search

function GREEDY-SEARCH (*problem*) **returns** a solution or failure
  return BEST-FIRST-SEARCH (*problem, h)*

h(n) = estimated cost of the cheapest path from the state at node n to a goal state



Greedy Search…



Not Optimal
Incomplete
$O(b^m)$ time
$O(b^m)$ space

Stages in a greedy search for Bucharest, using the straight-line distance to Bucharest as the heuristic function h_{SLD}. Nodes are labeled with their h-values.

Beam Search

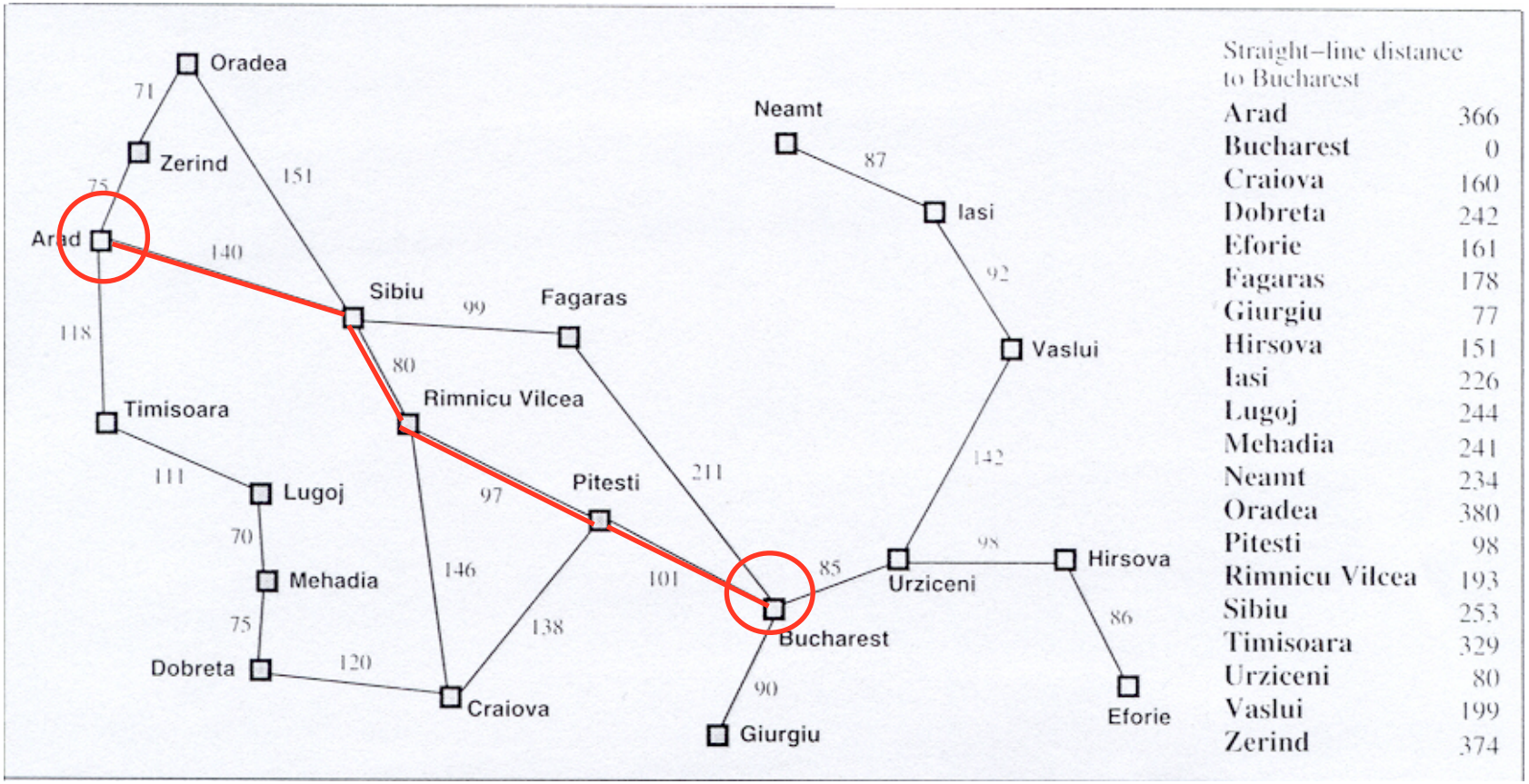
Use $f(n) = h(n)$ but $|\text{nodes}| \leq K$

- Not complete
- Not optimal

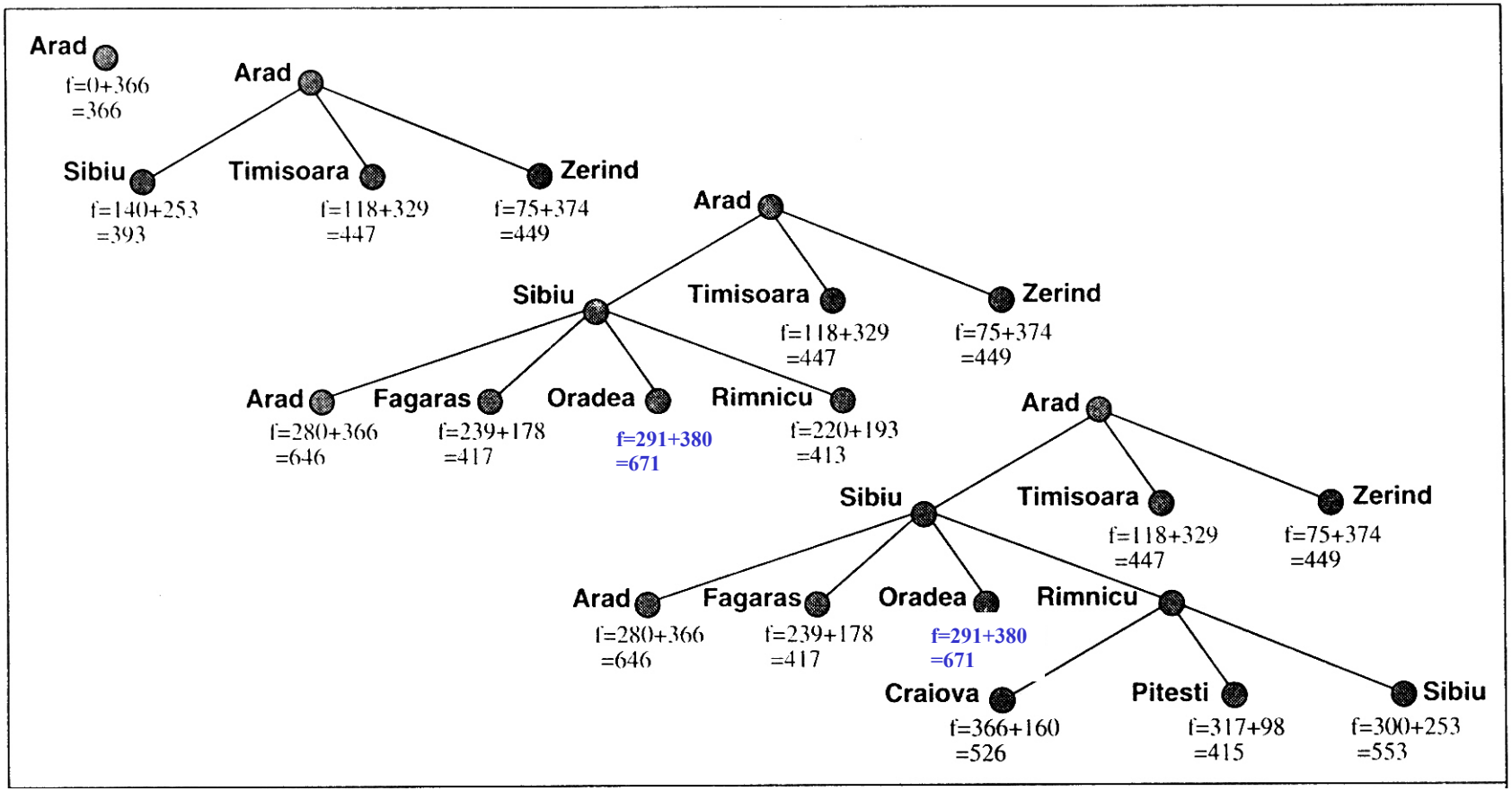
A* Search

function A*-SEARCH (*problem*) **returns** a solution or failure
return BEST-FIRST-SEARCH (*problem, g+h)*

f(n) = estimated cost of the cheapest solution through n
= g(n) + h(n)



A* Search…



Stages in an A* search for Bucharest. Nodes are labeled with $f = g + h$. The h values are the straight-line distances to Bucharest.

A* Search…

In a minimization problem, an admissible heuristic h(n) *never overestimates the real value*

(In a maximization problem, h(n) is admissible if it never *under*estimates)

Best-first search using f(n) = g(n) + h(n) and an admissible h(n) is known as *A* search*

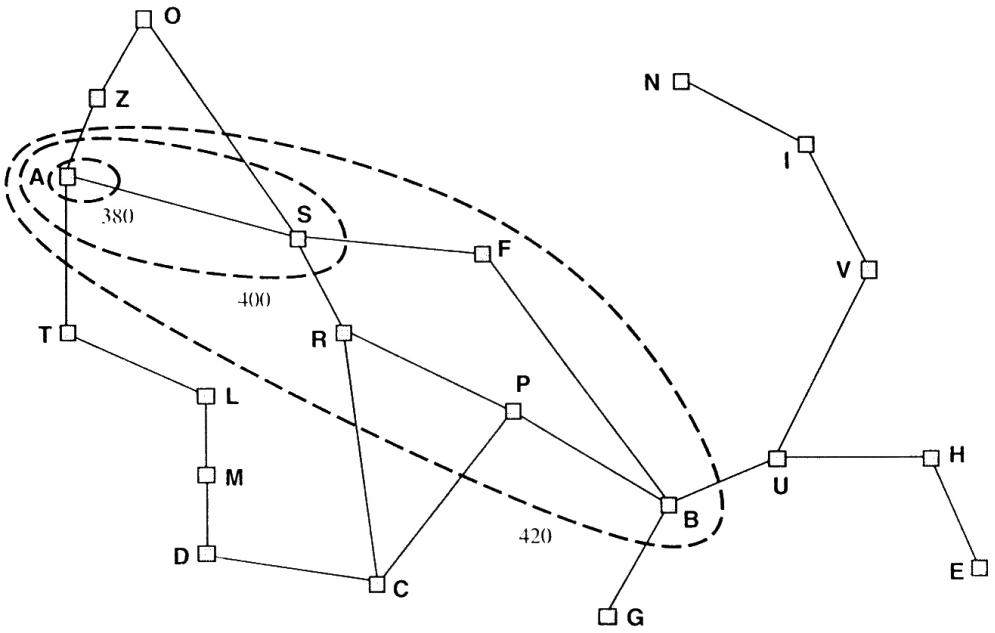
A* tree search is complete & optimal

Monotonicity of a heuristic

h(n) is monotonic if, for every node n and every successor n' of n generated by any action a, the estimated cost of reaching the goal from n is no greater than the step cost of getting to n' plus the estimated cost of reaching the goal from n' : $h(n) \leq c(n,a,n') + h(n')$.

This implies that f(n) (which equals g(n)+h(n)) never decreases along a path from the root.
Monotonic heuristic => admissible heuristic.

With a monotonic heuristic, we can interpret A* as searching through contours:



Map of Romania showing contours at $f = 380$, $f = 400$ and $f = 420$, with Arad as the start state. Nodes inside a given contour have f-costs lower than the contour value.

Monotonicity of a heuristic…

A* expands all nodes n with f(n) < f*, and may expand some nodes right on the "goal contour" (f(n) = f*), before selecting a goal node.

With a monotonic heuristic, even A* *graph* search (i.e., search that deletes later-created duplicates) is optimal.

Another option, which requires only admissibility – not monotonicity – is to have the duplicate detector always keep the best (rather than the first) of the duplicates.

Completeness of A*

Because A* expands nodes in order of increasing f, it must eventually expand to reach a goal state. This is true unless there are infinitely many nodes with $f(n) \leq f^*$

How could this happen?

- There is a node with an infinity branching factor
- There is a path with finite path cost but an infinite number of nodes on it

So, A* is complete on graphs with a finite branching factor provided there is some positive constant δ s.t. every operator costs at least δ

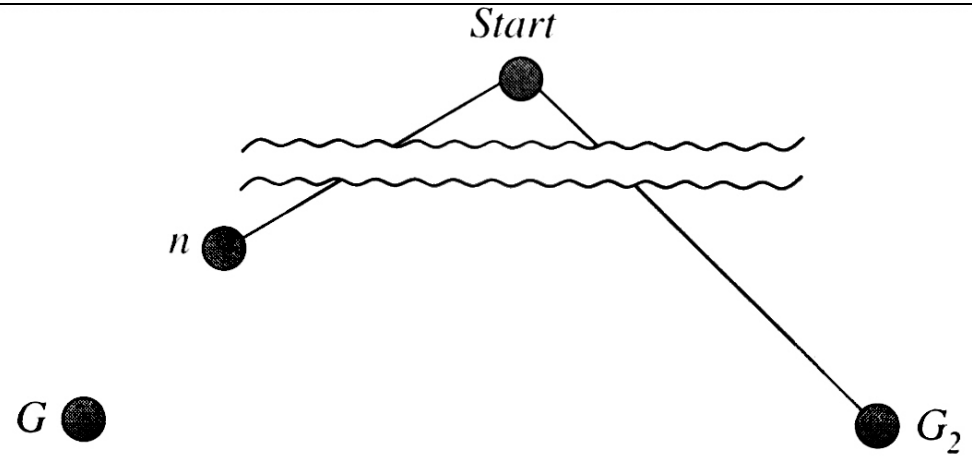
Proof of optimality of A* *tree* search

Assumes h is admissible, but does not assume h is monotonic

Let G be an optimal goal state, and f(G) = f* = g(G).
Let G_2 be a suboptimal goal state, i.e. $f(G_2) = g(G_2) > f^*$.
Suppose for contradiction that A* has selected G_2 from the queue. (This would terminate A* with a suboptimal solution)

Let n be a node that is currently a leaf node on an optimal path to G.

Situation at the point where a sub-optimal goal state G_2 is about to be picked from the queue



Because h is admissible, $f^* \geq f(n)$.

If n is not chosen for expansion over G_2, we must have $f(n) \geq f(G_2)$
So, $f^* \geq f(G_2)$. Because $h(G_2)=0$, we have $f^* \geq g(G_2)$, contradiction. □

Complexity of A*

- Generally $O(b^d)$ time and space.
- Sub-exponential growth when $|h(n) - h^*(n)| \leq O(\log h^*(n))$
  - Unfortunately, for most practical heuristics, the error is at least proportional to the path cost

A* is optimally efficient

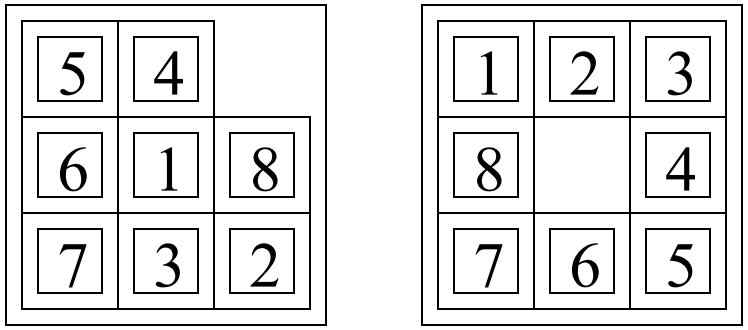
A* is optimally efficient for any given h-function among algorithms that extend search paths from the root. I.e. no other optimal algorithm is guaranteed to expand fewer nodes (for a given search formulation)

Intuition: any algorithm that does not expand all nodes in the contours between the root and the goal contour runs the risk of missing the optimal solution.

Generating heuristics (h-functions)

Heuristics (h(n)) for A*

A typical instance of the 8-puzzle



Start state Goal state

Heuristics?

h_1: #tiles in wrong position

h_2: sum of Manhattan distances of the tiles from their goal positions

h_2 dominates h_1: $\forall n,\ h_2(n) \geq h_1(n)$

Heuristics (h(n)) for A* …

	Search Cost			Effective Branching Factor		
d	IDS	A*(h_1)	A*(h_2)	IDS	A*(h_1)	A*(h_2)
2	10	6	6	2.45	1.79	1.79
4	112	13	12	2.87	1.48	1.45
6	680	20	18	2.73	1.34	1.30
8	6384	39	25	2.80	1.33	1.24
10	47127	93	39	2.79	1.38	1.22
12	364404	227	73	2.78	1.42	1.24
14	3473941	539	113	2.83	1.44	1.23
16	–	1301	211	–	1.45	1.25
18	–	3056	363	–	1.46	1.26
20	–	7276	676	–	1.47	1.27
22	–	18094	1219	–	1.48	1.28
24	–	39135	1641	–	1.48	1.26

Comparison of the search costs and effective branching factors for the ITERATIVE-DEPENING-SEARCH and A* algorithms with $h1$, $h2$. Data are averaged over 100 instances of the 8-puzzle, for various solution lengths.

It is always better to use a heuristic h(n) with higher values, as long as it does not overestimate.

<= A* expands all nodes with f(n) < f*

Inventing heuristic functions h(n)

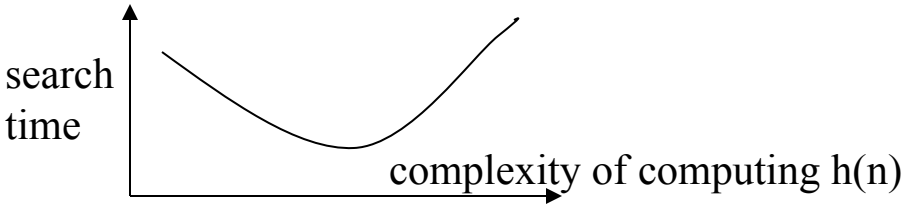
Cost of exact solution to a relaxed problem is often a good heuristic for original problem.
Relaxed problem(s) can be generated automatically from the problem description by dropping or relaxing constraints.
Most common example in operations research: relaxing all integrality constraints and using linear programming to get an optimistic h-value.

What if no dominant heuristic is found?
$h(n) = \max [h_1(n), \ldots h_m(n)]$
h(n) is still admissible & dominates the component heuristics

Use probabilistic info from statistical experiments: "If h(n)=14, h*(n)=18".
Gives up optimality, but does less search

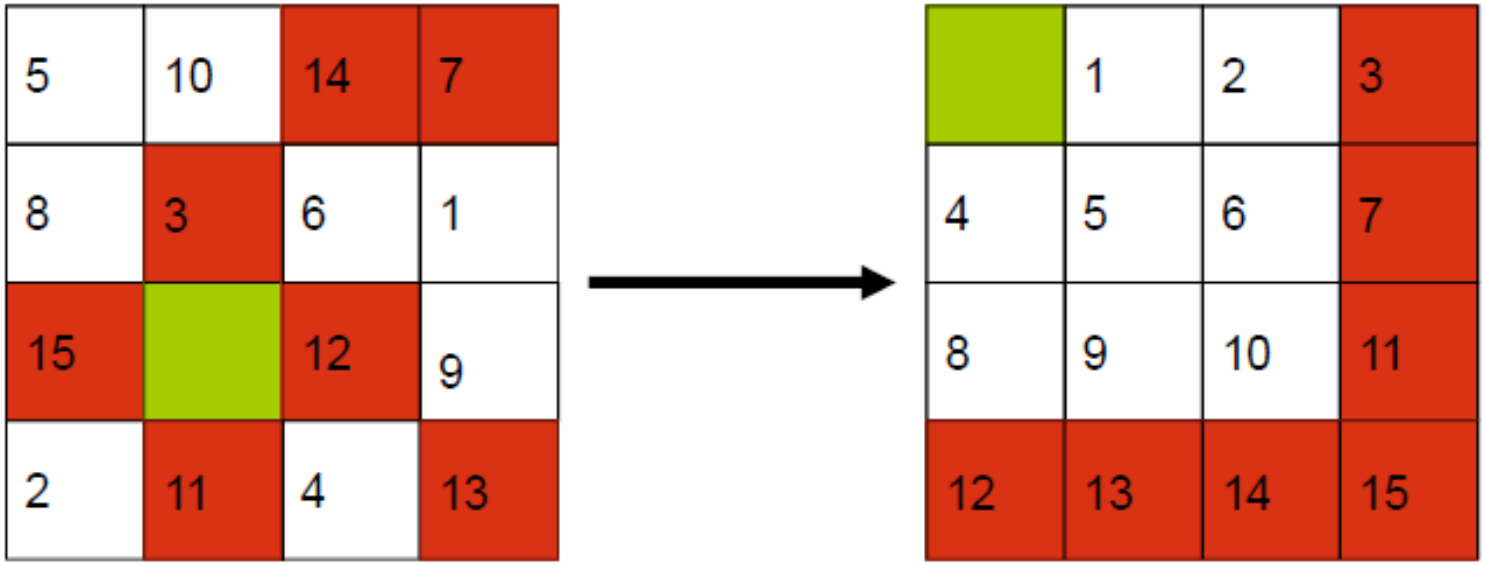
Pick features & use machine learning to determine their contribution to h.

Use full breath-first search as a heuristic?



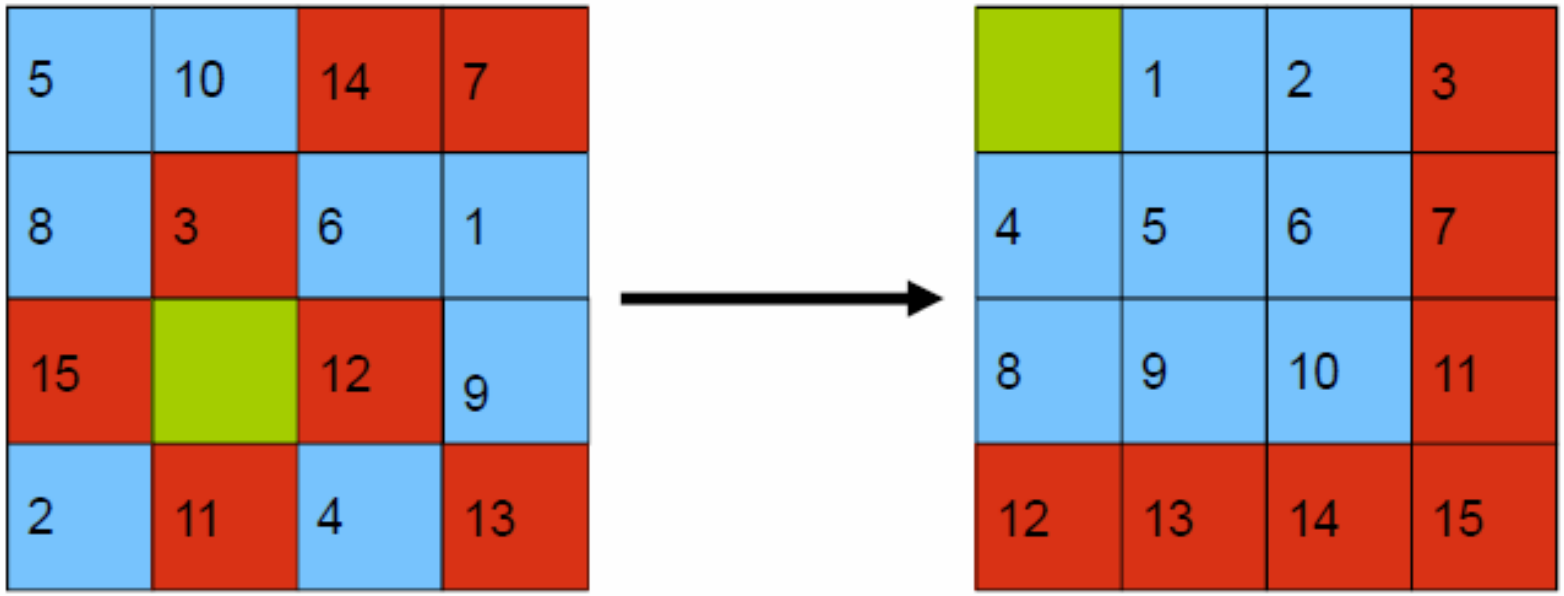
PATTERN DATABASES FOR HEURISTICS

[Culberson & Schaeffer 1996; many improvements since; figures in this segment borrowed from others' AI courses]



31 moves is a lower bound on the total number of moves needed to solve this particular state.

Combining multiple pattern databases



31 moves needed to solve red tiles

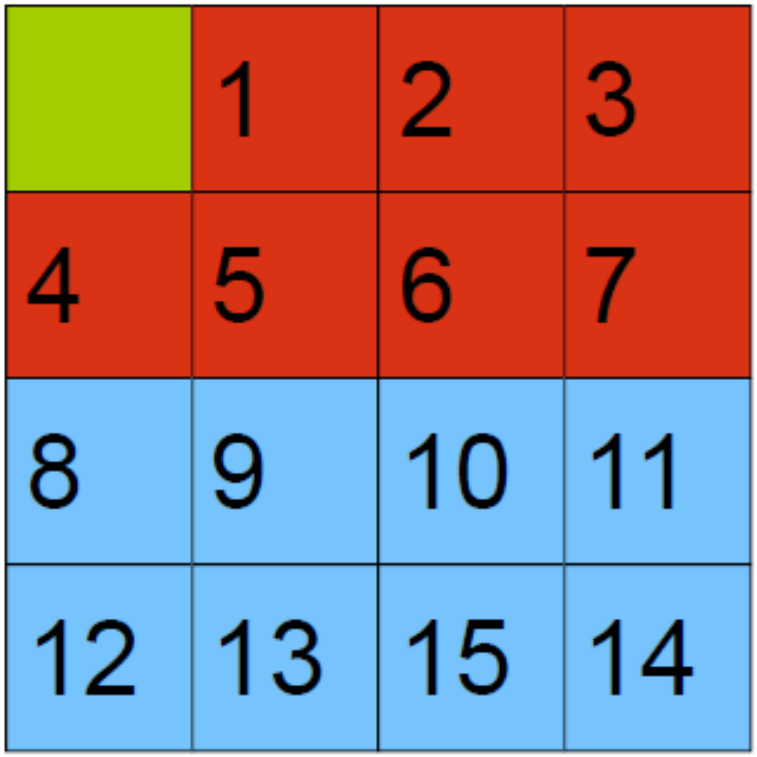
22 moves need to solve blue tiles

Overall heuristic is maximum of 31 moves

Additive pattern databases

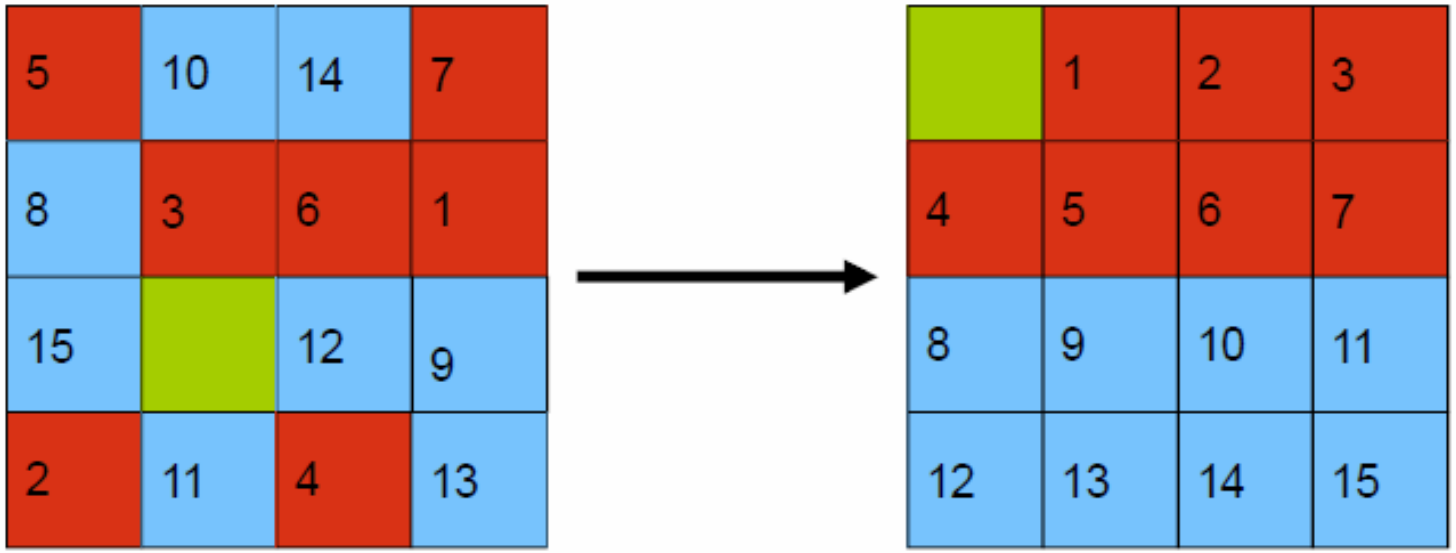
- Original pattern databases counted all moves to get pattern pieces into their correct places
- *Additive pattern databases* count only moves of the pieces belonging to the pattern
  - Manhattan distance is a special case where each piece is its own pattern
- Then, can *add* the values of the different patterns instead of taking the *max*

Example additive pattern database



The 7-tile database contains 58 million entries. The 8-tile database contains 519 million entries.

Computing the heuristic value in an additive pattern database



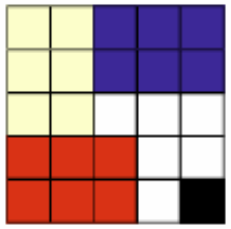
20 moves needed to solve red tiles

25 moves needed to solve blue tiles

Overall heuristic is sum, or 20+25=45 moves

Performance

- 15 Puzzle: 2000x speedup *vs* Manhattan dist
  - IDA* with the two DBs shown previously solves 15 Puzzles optimally in 30 milliseconds
- 24 Puzzle: 12 million x speedup *vs* Manhattan
  - IDA* can solve random instances in 2 days.
  - Requires 4 DBs as shown
    - Each DB has 128 million entries
  - Without PDBs: 65,000 years



More efficient variants of A* search (and approximations)

Approximate A* versions

(less search effort, but only an approximately optimal solution)

1. Dynamic weighting:
   $f(n) = g(n) + h(n) + \alpha [1 - (depth(n) / N)] h(n)$
   where N is (an upper bound on) depth of desired goal.
   Idea: Early on in the search do more focused search.
   Thrm: Solution is within factor $(1+ \alpha)$ of optimal.

2. A_β*:
   From open list, choose among nodes with f-value within a factor $(1+\beta)$ of the most promising f-value.
   Thrm: Solution is within factor $(1+\beta)$ of optimal
   Make the choice based on which of those nodes leads to lowest search effort to goal (sometimes picking a node with best h-value accomplishes this)

Best-bound search = A* search but uses tricks in practice

- A* search was invented in the AI research community
- *Best-bound search* is the same thing, and was independently invented in the operations research community
  - With heavy-to-compute heuristics such as LP, in practice, the commercial mixed integer programming solvers:
    - Use parent's f value to queue nodes on the open list, and only evaluate nodes exactly when (if) they come off the open list
      - => first solution may not be optimal
      - => need to continue the search until all nodes on the open list look worse than the best solution found
    - Do *diving*. In practice there is usually not enough memory to store the LP data structures with every node. Instead, only one LP table is kept. Moving to a child or parent in the search tree is cheap because the LP data structures can be incrementally updated. Moving to another node in the tree can be more expensive. Therefore, when a child node is almost as promising as the most-promising (according to A*) node, the search is made to proceed to the child instead.
      - Again, need to continue the search until all nodes on the open list look worse than the best solution found

Memory-bounded search algorithms

Iterative Deepening A* (IDA*)

```
function IDA*(problem) returns a solution sequence
  inputs: problem, a problem
  static: f-limit, the current f-COST limit
          root, a node

  root ← MAKE-NODE(INITIAL-STATE[problem])
  f-limit ← f-COST(root)
  loop do
      solution, f-limit ← DFS-CONTOUR(root,f-limit)
      if solution is non-null then return solution
      if f-limit = ∞ then return failure; end

function DFS-CONTOUR(node,f-limit) returns a solution sequence and a new f-COST limit
  inputs: node, a node
          f-limit, the current f-COST limit
  static: next-f, the f-COST limit for the next contour, initially ∞

  if f-COST[node] > f-limit then return null, f-COST[node]
  if GOAL-TEST[problem](STATE[node]) then return node, f-limit
  for each node s in SUCCESSOR(node) do
      solution, new-f ← DFS-CONTOUR(s,f-limit)
      if solution is non-null then return solution, f-limit
      next-f ← MIN(next-f, new-f); end
  return null, next-f
```

f-COST[*node*] = g[*node*] + h[*node*]

IDA* …

Complete & optimal under same conditions as A*.
Linear space. Same O( ) time complexity as A*.

If #nodes grows exponentially, then asymptotically optimal space.

IDA* …

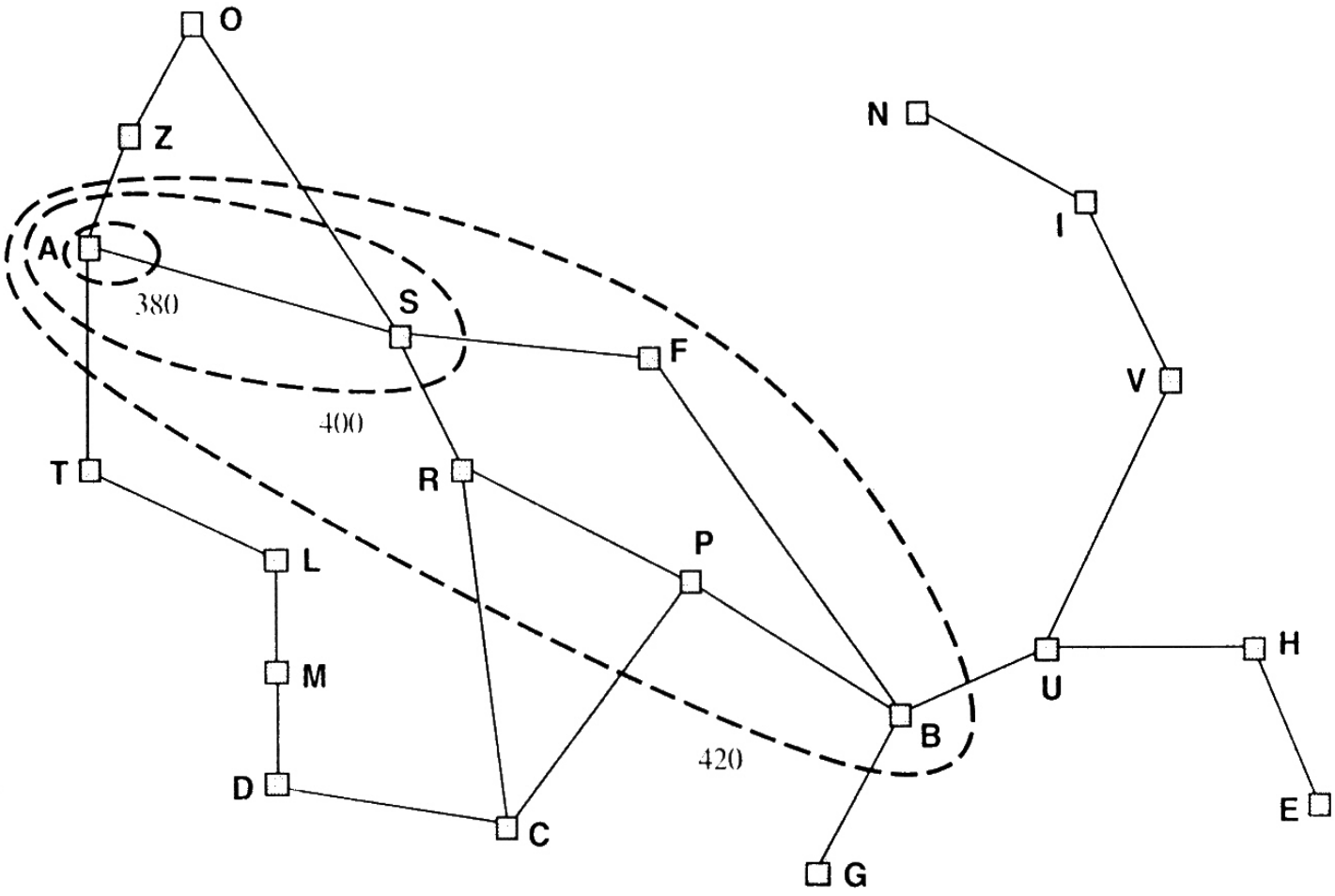
Effective e.g. in 8-puzzle where f typically only increases 2-3 times → 2-3 iterations.

Last iteration ~ A*

Ineffective in e.g. TSP where f increases continuously
→ each new iteration only includes one new node.

- If A* expands N nodes, IDA* expands $O(N^2)$ nodes
- Fixed increment ε → $\sim 1/\varepsilon$ iterations
- Obtains ε-optimal solution if terminated once first solution is found
- Obtains an optimal solution if search of the current contour is completed

A* vs. IDA*



Map of Romania showing contours at $f = 380$, $f = 400$ and $f = 420$, with Arad as the start sate. Nodes inside a given contour have f-costs lower than the contour value.

Memory-bounded search algorithms

- IDA* 1985
- Recursive best-first search 1991

} use too little memory

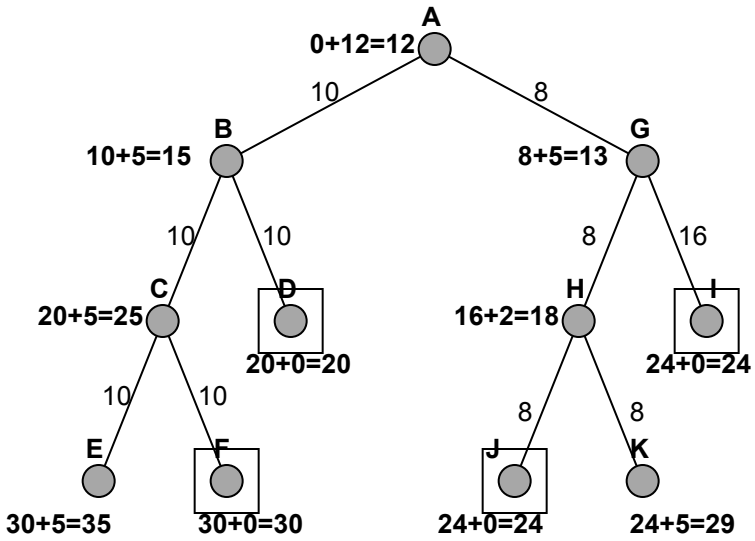
- Memory-bounded A* (MA*) 1989
- Simple memory-bounded A* (SMA*) 1992 …

Simple Memory-bounded A* (SMA*)

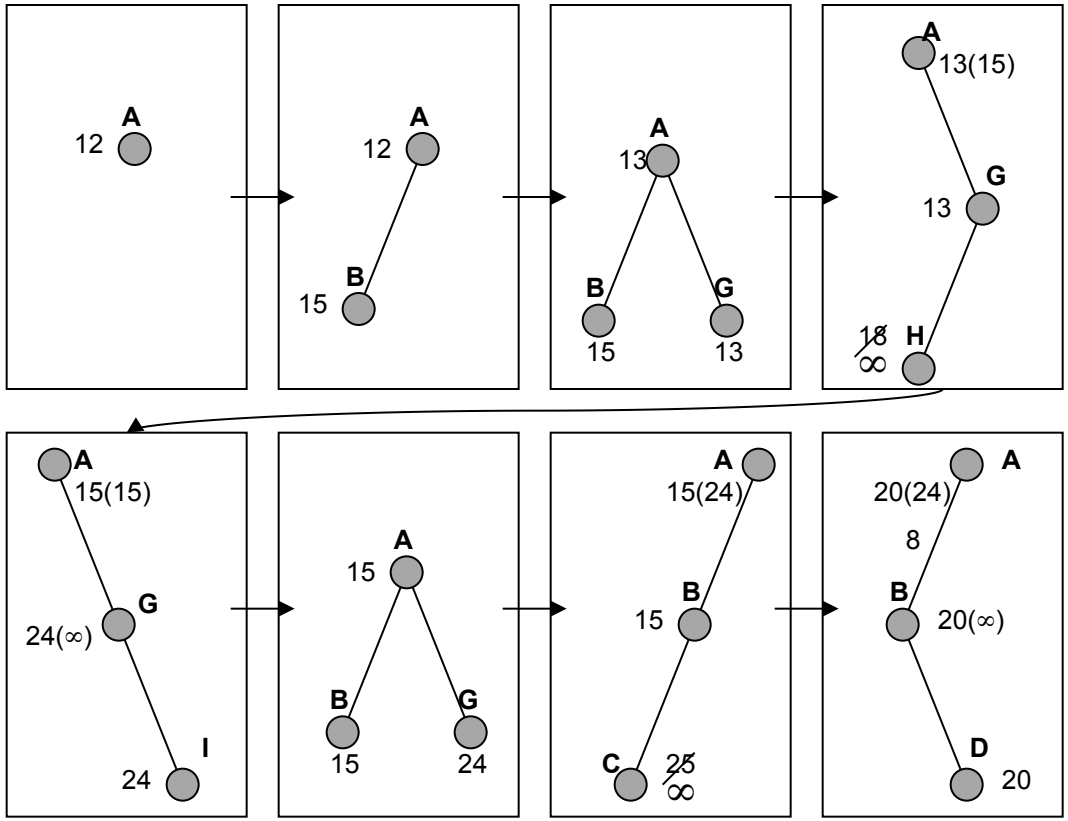
(Example with 3-node memory)

Search space

f = g+h ? = goal



Progress of SMA*. Each node is labeled with its *current f*-cost.
Values in parentheses show the value of the best forgotten descendant.



Optimal & complete if enough memory

Can be made to signal when the best solution found might not be optimal

- E.g. if J=19

SMA* pseudocode (from 1st edition of Russell & Norvig)

Numerous details have been omitted in the interests of clarity

```
function SMA*(problem) returns a solution sequence
  inputs: problem, a problem
  static: Queue, a queue of nodes ordered by f-cost

  Queue ← MAKE-QUEUE({MAKE-NODE(INITIAL-STATE[problem])})
  loop do
      if Queue is empty then return failure
      n ← deepest least-f-cost node in Queue
      if GOAL-TEST(n) then return success
      s ← NEXT-SUCCESSOR(n)
      if s is not a goal and is at maximum depth then
         f(s) ← ∞
      else
         f(s) ← MAX(f(n),g(s)+h(s))
      if all of n's successors have been generated then
         update n's f-cost and those of its ancestors if necessary
      if SUCCESSORS(n) all in memory then remove n from Queue
      if memory is full then
         delete shallowest, highest-f-cost node in Queue
         remove it from its parent's successor list
         insert its parent on Queue if necessary
      insert s in Queue
   end
```