day 16 3/9/98 "To be perverse, what we'll do now is..."

# 1 Max Flow

## 1.1 definitions

- flow-network = directed $G = (V, E)$
- Capacities $C(u, v) \geq 0$ If $(u, v) \notin E \Rightarrow c(u, v) = 0$
- For $s, t \in V$ let $s = source$, $t = sink$
- flow = $f : V \times V \rightarrow R$ s.t.

1. $f(u, v) \leq c(u, v)$
2. $f(u, v) = -f(v, u)$
3. $u \in V - \{s, t\}$ s.t. $\sum_{v \in V} f(u, v) = 0$

- $|f| = \sum_{v \in V} f(s, v)$ = net flow

The max flow problem has as input a flow network and as output a flow, $f$, with maximum net flow.

# 2 Example problems

## 2.1 bananas

There are a set of banana producers and a set of banana consumers. We want to maximize the number of bananas succesfully fed to consumers. Make the source node connect to every banana producer with capacity = banana production quantity. Make the sink node connect to every consumer with capacity = number of bananas eaten. Connect producers to consumers with capacity = amount of shipping capacity. Solve the problem. The solution maximizes shipping consumption.

## 2.2 network flow with delays

Nodes are connected with some capacity and some delay. We want to mazimize the flow within some amount of time, $t$. Expand the graph to $G \times t = (V \times t, E_t)$. Two nodes $v_{t_1}, u_{t_2}$ are connected if $(v, u) \in E$ and $t_2 - t_1$ = delay on that edge. Solving this flow problem will give the solution.

# 3 Algorithm

## 3.1 definitions

- residual capacity: $f$ = flow on $G$, $C_f(u,v) = C(u,v) - f(u,v)$
- residual network: $E_f = \{(u,v) \in V^2 | C_f(u,v) > 0\}$

Given the residual network, we need to find an "augmenting path" which increases the flow.

- Partition $A, B$ of $V$ is $s, t$ cut if $s \in A, t \in B$.
- Cost of the cut: $C(A,B) = \sum_{u \in A, v \in B} c(u,v)$
- In general: $f(A,B) = \sum_{u \in A, v \in B} f(u,v)$

## 3.2 lemma

if $f$ a flow on $G$, $G_f$ is a residual then

1. $f'$ a flow on $G_f$ iff $f + f'$ a flow on $G$
2. $f'$ max flow on $G_f$ iff $f + f'$ is a max flow on $G$
3. $|f + f'| = |f| + |f'|$

Proof of (1) $f'$ is flow on $G_f$ then $f'(e) \leq C_f(e) = C(e) - f(e)$ Which implies $f'(e) + f(e) \leq C_f(e)$

## 3.3 Another lemma

$\forall s, t$ cuts $A, B$ $f(A,B) = |f|$.

## 3.4 Yet another lemma

$|f| \leq C(A,B)$

## 3.5 Max flow min cut theorem

The following are equivalent:

- $f$ is a max flow
- $\exists$ cut $A, B$ s.t. $C(A,B) = |f|$
- There does not exist an augmenting path

### 3.6 Algorithm

1. Find augmenting path -> none means you are done
2. Calculate residual
3. repeat

Add up all the augmenting paths to get the flow.

# 4 Analysis

The convergence time can be very long. For integers it can be at least proportional to $|f|$. If you allow noninteger capacities the algorithm may not even be convergent.