1 Shortest path algorithms

- G is a digraph
- $w(e)$ = weight of edge e
- P is a path
- $w(P) = \sum_{i=1}^{k-1} w(v_i \rightarrow v_{i+1})$
- $\delta(u, v) = min_{P:u\rightarrow v} w(P)$ = weight of minimum weight path from u to v

There are several categories of weights:

- $w(e) = 1$ always (weights always the same)
- $w(e) \geq 0$ always (weights always positive)
- $w(e) \in R$ (weights have any value)

In addition, there are 2 basic flavors of the algorithms

- single pair or single source
- all pairs

Here is a table of the algorithms which are used.

	$w(e) = 1$	$w(e) \geq 1$	$w(e) \in R$
$Single$	$BreadthFirstSearch$	$Dijkstra$	$Bellman - Ford$
all	BFS	$Dijkstra$	$Johnson$

2 Rings

A ring is defined by $R = (S, \bigoplus, \bigotimes, 0, 1)$ where S is a set, $\bigotimes, \bigoplus : S^2 \rightarrow S$ are binary operations. 0 is the identity element of $\bigoplus$ and 1 the identity element of $\bigotimes$. Rings also obey the following rules:

- $(a \bigoplus b) \bigoplus c = a \bigoplus (b \bigoplus c)$ and $(a \bigotimes b) \bigotimes c = a \bigotimes (b \bigotimes c)$ (associativity)
- $a \bigoplus b = b \bigoplus a$ (commutativity)
- $(a \bigoplus b) \bigotimes c = (a \bigotimes c) \bigoplus (b \bigotimes c)$ and $a \bigotimes (b \bigoplus c) = (a \bigotimes b) \bigoplus (a \bigotimes c)$ (distributivity)
- $a \bigoplus 0 = 0 \bigoplus a = a$
- $a \bigotimes 1 = 1 \bigotimes a = a$
- $\exists (-a) : a \bigoplus (-a) = 0$ (inverse)

A semiring must satisfy every requirement except for the last.

2.1 ring matrices

Given any ring a matrix ring, $R^{n\times n} = (S^{n\times n}, \bigoplus^{n\times n}, \bigotimes^{n\times n}, 0^{n\times n}, 1^{n\times n})$ can be defined. $0^{n\times n}$ is the 0 matrix and $1^{n\times n}$ is the identity matrix. The addition operator is defined elementwise and the multiplication operator is defined as: $A^{n\times n} \bigotimes^{n\times n} B^{n\times n} = \bigoplus_{k=1}^{n}(A_{ik} \bigotimes B_{kj})$.

2.2 ring examples

- Minimum weight paths can be defined using a ring $(R\cup\{\infty\}, min, +, \infty, 0)$.
- Another ring: $(R^+ \cup \{\infty\}, max, *, 0, 1)$.
- boolean ring: $(\{0,1\}, \bigvee, \wedge 0, 1)$
- $(Z \cup \{\pm\infty\}, min, max, +\infty, -\infty)$

2.3 Shortest path through rings

Consider the $min, +$ ring above. Construct an $n \times n$ ring from it. Assign the weights, $A_{ij} = \begin{array}{ccc} w_{ij} & if & (v_i, v_j) \in E \\ 0 & if & i = j \\ \infty & else & \end{array}$

Claim: $(A^k)_{ij} = min_{P:\#P\leq k}\, w(P)$ **(the ij entry is the minimal path for all paths of length less than k)** To compute all pairs shortest paths, calculate A^{n-1}.

2.4 timing analyses

The simplest way to calculate this is by multiply A together $n-1$ times which has cost $O(n^4)$. This can be cut to $O(n^3 log(n))$ by doing the multiplications in a more clever order.