Random Walks on Graphs

15-750
4/17/19

Graph: $G = (V, E, w)$ (possibly directed)

$$w_i = w(V_i) \equiv \sum_{(i,j)\in E} w_{ij} \qquad P_{ij} \equiv w_{ij}/w_i$$

Random walk on G:

Suppose at a given time we are at $v_i \in V$.

We move to V_j with probability P_{ij}

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e.g. $V \equiv$ all ordings of a deck of 52 cards

$P_{ij} \equiv$ Prob of going from order_i to order_j in one shuffle.

Question: Why do professionals play after 5 shuffles?

Two views of a random walk

1) Particle view (def)
2) Wave, Probability dist, or large # of simultaneous of independent walkers

for 2) $\chi^{(i)}$ = dist at time i then

$$AD^{-1}\chi^{(i)} = \chi^{(i+1)}$$

$A \equiv$ adj matrix

$D \equiv$ dia degree

Important Parameters

Access time: or Hitting time

H_{ij} = Expected time to visit j starting at i

Commute Time:

$$K(i,j) = H(i,j) + H(j,i)$$

Cover Time:

Expected time to visit all nodes

max over all starting nodes

Mixing Rate: (not do)

Random Walks – the Symmetric Case

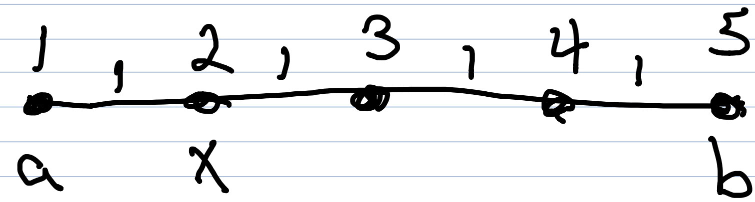
Idea: View a random walk as a walk on a network of conductors!

Input: $G=(V,E,c)$ $\quad c_{ij}=c_{ji}$

Consider a random walk starting at x and ending at b.

<u>Def</u> $h_x \equiv$ prob we visit a before b starting at x, $a \neq b$.

eg

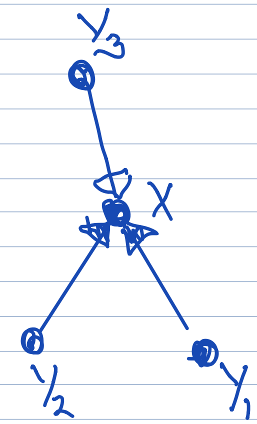


$h_a = 1$ & $h_b = 0$

h_2? $\quad h_2 > 1/2$ why?

$h_a = 1$ & $h_b = 0$

Suppose $x \neq a, b$



<u>Claim</u> $h_x = \sum_y P_{xy} h_y$

$P_{xy} \geq 0$ & $\sum_y P_{xy} = 1$

$\therefore$ h_x is a convex comb of its neighbors!

h is harmonic with bdary a, b!

Lets construct an identical electrical prob!

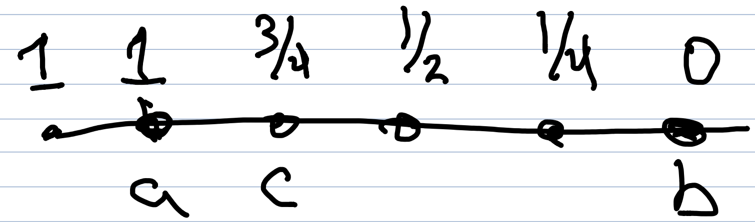
Consider: $V_a = 1$ & $V_b = 0$

$\forall x \neq a, b \quad V_x = \sum_y \frac{C_{xy}}{C_x} V_y$

but $C_{xy}/C_x = P_{xy}$! $\Rightarrow h = V$

<u>Thm</u> Set $V_a = 1$ & $V_b = 0$; Let $x \neq a, b$ "float" then $V_x =$ prob visit a before b. Residual current at $x = 0$.

e.g.

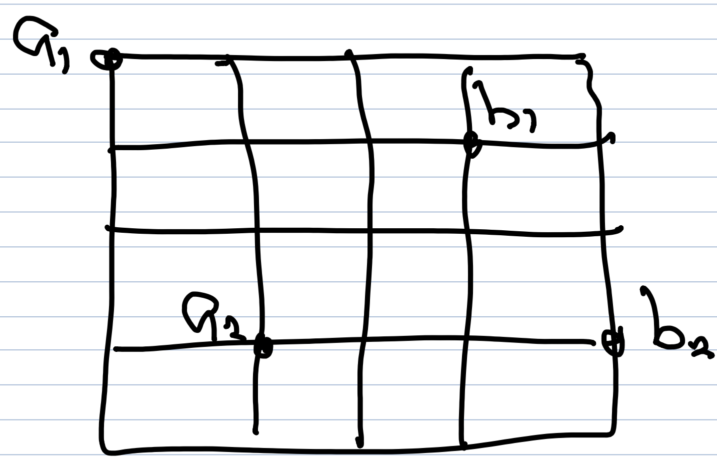


$h_c = 3/4$

$a = x_1$ & $b = x_n$

Algebraically: $L \begin{pmatrix} 1 \\ * \\ \vdots \\ * \\ 0 \end{pmatrix} = \begin{pmatrix} * \\ 0 \\ \vdots \\ 0 \\ * \end{pmatrix}$

In general may have multi sinks and goals.



Thus computable in one Laplacian solve.

Interpretation of Current for Random Walk.

Consider 1 unit of potential current flow from a to b, say i

what does i_{xy} correspond to in random walk from a to b?

<u>Thm</u> i_{xy} = Expected net # of traversals of E_{xy} in a random walk from a to b.

(No proof)

How to compute hitting time

Def $h(x,b) \equiv$ expected time to reach b from x.

$h_x = h(x,b)$ b fixed

Lets write a recurrence:

$$h_b = 0$$

$$x \neq b \quad h_x = 1 + \sum_y h_y P_{xy}$$

Let's think of h_x as voltage V_x

$$V_b = 0 \quad V_x = 1 + \sum_y \frac{C_{xy}}{C_x} V_y$$

$$C_x V_x = C_x + \sum_y C_{xy} V_y$$

$$\underbrace{C_x V_x - \sum C_{xy} V_y}_{\text{Graph Laplacian}} = \underbrace{C_x}_{\text{residual current}}$$

n-1 constraints

by adding constraint for $V_n = b$

$$\underset{V_n=0}{LV} = \begin{pmatrix} c_1 \\ \vdots \\ c_{n-1} \\ \delta \end{pmatrix} \qquad \begin{aligned} b &= V_n \\ c &= \sum c_i \end{aligned}$$

where $\delta = c_n - c$

<u>Alg</u> for hitting time to V_n.

solve $\underset{V_n=0}{LV} = \begin{pmatrix} c_1 \\ \vdots \\ c_n - c \end{pmatrix}$ return V_x

What about commute time?

For $v_1 = a$ & $v_n = b$.

Solution 1

solve $LV^b = \begin{pmatrix} c_1 \\ \vdots \\ c_n - c \end{pmatrix}$ $LV^a = \begin{pmatrix} c_1 - c \\ \vdots \\ c_n \end{pmatrix}$

$h(1,n) = V_1^b - V_n^b$ & $h(n,1) = V_n^a - V_1^a$

set $V = V^b - V^a$

$C(1,n) = (V^b - V^a)_1 - (V^b - V^a)_n = V_1 - V_n$

Solution 2

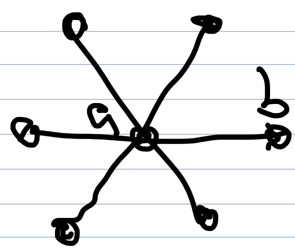
$L(V^b - V^a) = LV^b - LV^a =$

$$\begin{pmatrix} c_1 \\ \vdots \\ c_n - c \end{pmatrix} - \begin{pmatrix} c_1 - c \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} c \\ 0 \\ \vdots \\ 0 \\ -c \end{pmatrix} = c \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}$$

Solve $LV = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}$

return $C(V_1 - V_n)$ but $(V_1 - V_n) = ER_{1n}$

Thm $C(a,b) = C \cdot ER_{ab} = 2m\, ER_{ab}$



$C(a,b) = 2(n-1)$.