15-750
4/29/19

Parallel Maximal Independent Set

Luby's Algorithm

Input: Graph $G=(V,E)$ (undirected)

Def $I \subseteq V$ in independent if no pair share an edge.

An Ind set I is

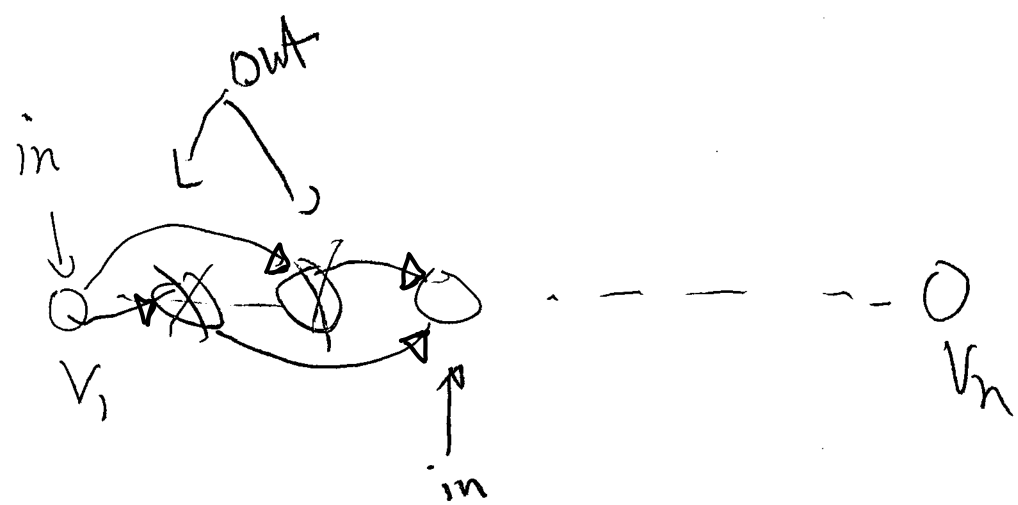
maximal if $\forall w \in V \setminus I$ $w \cup I$ in not ind.

Maximum if $\forall$ ind sets I' $|I| \geq |I'|$

Note Finding a maximum ind set is NP-Hard.

The simple greedy alg computes a maximal ind in $O(m+n)$ time.

<u>Note</u> Computing this ind set is P-Complete (The lexiographically first MIS)



<u>Note</u> Our List-Rank & Parallel-Tree alg were of following form

1) Find a large independent set of nodes.
2) Process them (remove them)

<u>Def</u> MIS $\equiv$ Maximal Ind Set of G.

Goal: Parallel MIS Alg!

We consider Alg of form for $G=(V,E)$

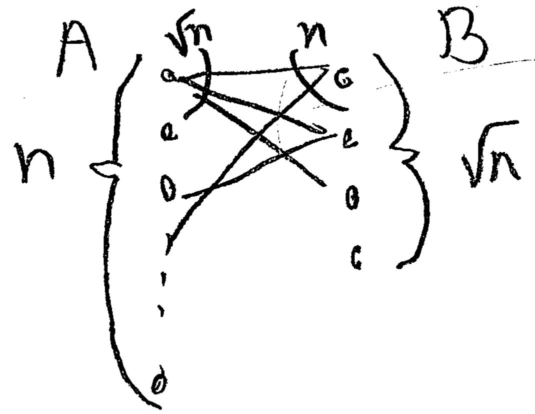
1) Each node v tries to join with prob(v).
2) Break ties giving set $I \subseteq V$.
3) Remove I & Neig(I) and all adj edges.
4) Repeat

Suppose $\text{Prob}(v) \approx \frac{1}{d_v}$ $d_v \equiv$ degree of v.

<u>The hope</u> After each round $|V'| \le c|V|$ $c<1$

Bad Example:

Input: $K_{n,\sqrt{n}}$



$$\text{Expect}(|I \cap A|) \approx n\left(\frac{1}{\sqrt{n}}\right) \approx \sqrt{n}$$

$$\text{Expect}(|I \cap B|) \approx \sqrt{n}\left(\frac{1}{n}\right) = \frac{1}{\sqrt{n}}$$

$$|I \cap B| \approx 0$$

Expect $|V'| \approx |V|$

But $E' \not\approx \emptyset$

<u>Note</u> Edges out of B caused $E' \approx 0$.

Randomized Parallel MIS

Init: $I = \emptyset$

While $E \neq \emptyset$ MIS$(G=(V,E))$

Proc MIS$(G=(V,E)$

0) Add all singeltons to I
1) In ll each v picks random $P(x) \in [0,1]$ uniformly
2) Vertex v joins I if $P(v) < P(\text{Neigh}(v))$
3) Set $V' = V \setminus (I \cup \text{Neigh}(I)$

   Set $E' = E \setminus \text{Edges}(I \cup \text{Neigh}(I))$
4) return $I \cup$ MIS$(G'=(V',E'))$

Note Step 1 is equivalent

1') Pick random permutation $\rho : V \to \{1, \dots, n\}$

Thus V joins I with prob $= \frac{1}{d_v + 1}$

Implementation Details

0) Assign a processor to each vertex and edge.

1) Each vertex proc generate $P(v)$.

2) [Computing I]

   a) Each edge (u,v) reads $P(u)$ & $P(v)$ (CR)

   b) It marks $P(u)$ if $P(u) > P(v)$. (CW)

   c) Vertex joins I if unmarked

3) Each edge (u,v) marks u for removal if $v \in I$.

4) Each edge (u,v) with removed endpoint removes itself.

5) Use Prescan to pack G'.

Consider Random Variables

$Y = \#$ edges removed

$$Y_{uv} = \begin{cases} 1 & \text{if } uv \in E \text{ is removed} \\ 0 & \text{o.w.} \end{cases}$$

Thus $2Y = \sum_{uv \in E} Y_{uv}$

<u>Main Thm</u> $2\,\mathbb{E}[Y] \geq |E|$

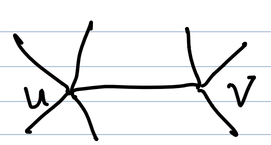
ie Expected # edges removed $\geq |E|/2$.

Critical Random Variable: $uv \in E$

$$X_{uv} = \begin{cases} 1 & \text{if } P(u) < P(\text{Neig}(u), \text{Neig}(v) \setminus u) \\ 0 & \text{o.w.} \end{cases}$$

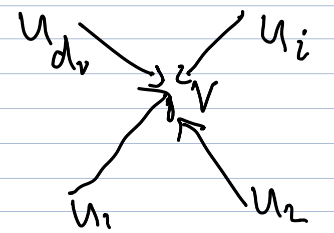
Note: $X_{uv} = 1 \Rightarrow u \in I$ and v is out.

<u>Claim</u> $\mathbb{E}[X_{uv}] = \text{Prob}[X_{uv} = 1] \geq \frac{1}{d_u + d_v}$



<u>Claim A</u> $\sum_{uv \in E} d_v X_{uv} \leq 2Y$

Consider all edges into $v \in V$.



At most one $X_{u_i v} = 1$

ie $\sum_{u \in Neig(v)} X_{uv} \leq 1$

<u>On the otherhand</u>

$X_{u_i v} = 1 \Rightarrow Y_{u_j v} = 1$ for $\forall u_j \in Neig(v)$

<u>Thus</u> $\sum_{u \in Neig(v)} d_v X_{uv} \leq \sum_{u \in Neig(v)} Y_{uv}$

<u>Finally</u> $\sum_{uv \in E} d_v X_{uv} = \sum_{v \in V} \sum_{u \in N(v)} d_v X_{uv}$

$$= \sum_{v \in V} \sum_{u \in N(v)} Y_{uv} \leq \sum_{uv \in E} Y_{uv} = 2Y$$

$\square$

From Claim A we get

$$2Y \geq \sum_{\substack{(uv) \in E \\ \text{ordered} \\ \text{pairs}}} d_v X_{uv} = \sum_{\substack{(uv) \in E \\ \text{unordered} \\ \text{pairs}}} d_v X_{uv} + d_u X_{vu}$$

Taking expectation

$$2\,\mathbb{E}[Y] \geq \mathbb{E}\Big[\sum_{\substack{uv \in E \\ \text{unordered}}} d_v X_{uv} + d_u X_{vu} \Big]$$

$$= \sum_{uv \in E} d_v\, \mathbb{E}[X_{uv}] + d_u\, \mathbb{E}[X_{vu}]$$

$$\geq \sum_{uv \in E} \frac{d_v}{d_u + d_v} + \frac{d_u}{d_u + d_v} = \sum_{uv \in E} \frac{d_u + d_v}{d_u + d_v}$$

$$= \sum_{uv \in E} 1 = |E|$$

ie $|E| \leq 2\,\mathbb{E}[Y]$

Thm Random Parallel MIS terminates in $O(\log n)$ rounds with high prob.

Pf We use Chernoff bds.

Consider random variables

$$Z_i = \begin{cases} 1 & \text{if round } i \text{ deletes at least } 1/3 \text{ remaining edges.} \\ 0 & \text{o.w.} \end{cases}$$

Recall (Markov)

If $X \geq 0$ RV then $\text{Prob}[X \geq a] \leq \frac{\mathbb{E}(X)}{a}$

Consider $\bar{Y} = |E| - Y$ (edge not deleted)

$$\text{Prob}[Z_i = 0] = \text{Prob}\left[\bar{Y} \geq \tfrac{2}{3}|E|\right] \leq \frac{\mathbb{E}(\bar{Y})}{\tfrac{2}{3}|E|} \leq \frac{|E|/2}{\tfrac{2}{3}|E|} = 3/4$$

Thus $\text{Prob}[Z_i = 1] > 1/4$

Note Alg done if after r rounds $\sum_{i=1}^{r} Z_i \geq \log_{3/2} m$.

Claim [Chernoff Bds] After $8 \log_{3/2} m$ rounds

$$\text{Prob}\left[\sum_{i=1}^{r} Z_i < \log_{3/2} m\right] \leq n^{-\frac{1}{4}\log_{3/2} e} \qquad (*)$$

Pf Note the Z_i are not independent!

Chernoff bounds require independence!

Let $Y_i = \{0,1\}$ be iid RV $Prob[Y_i = 1] = 1/4$.

[Chernoff] $Prob\left[\sum_{i=1}^{r} Z_i < \log_{3/2} m\right] \le n^{-1/4 \log_{3/2} e}$

Intuitively Claim(*) should be true

Since $Prob[Z_i = 1] \ge Prob[Y_i = 1]$

Unfortunately Claim(*) needs a proof.

We will use Coupling Argument!

Let $G_0, G_1, \cdots, G_k$ be graphs generated by MIS

Let $P_i = Prob[Z_i = 1 \mid G_i]$ thus $P_i \ge 1/4$.

<u>Goal</u> Generate sequence of RV $X_1, \cdots, X_k$

st $Prob[Z_i = 1] = Prob[X_i = 1]$

Consider If $Y_i = 1$ then $X_i = 1$

If $Y_i = 0$ then $X_i = 1$ with prob $\frac{P_i - 1/4}{1 - 1/4}$

<u>Note</u> $Prob[X_i = 1] = 1/4 + \cancel{(1 - 1/4)}\left(\frac{P_i - 1/4}{\cancel{1 - 1/4}}\right) = P_i$

& $X_i \ge Y_i$

$Prob\left[\sum_{i=1}^{k} Z_i < \mu\right] = Prob\left[\sum_{i=1}^{k} X_i < \mu\right] \le Prob\left[\sum_{i=1}^{n} Y_i < \mu\right]$

$\square$