15-750

2/8/19

Dynamic Programming & Graphical Model

Graphical Models for prob. dist.

A way to define high dimensional prob dist.

Factor Graphs an example first

Ex. Medical diagnosis

$$C = \begin{cases} 1 & \text{if I have a cold} \\ 0 & \text{o.w.} \end{cases}$$

$$S = \begin{cases} 1 & \text{if I have a sore throat} \\ 0 & \text{o.w} \end{cases}$$

$$r = \begin{cases} 1 & \text{if I have a runny nose} \\ 0 & \text{o.w} \end{cases}$$

$$P = \begin{cases} 1 & \text{if there is pollen in the air} \\ 0 & \text{o.w.} \end{cases}$$

C	S	r	$\Psi_a(c,s,r)$
1	1	1	.1
0	0	1	.3
0	0	0	.6

r	P	$\Psi_b(r,p)$
0	1	.1
1	0	.2
1	1	.3
0	0	.4

Assume ψ_a & ψ_b ind unless told o.w.

ie $Prob(c=0, s=0, r=1, p=1) = (.3)(.3) = .09$

$$P(c,s,r,p) = \frac{1}{Z}\psi_a(c,s,r)\,\psi_b(r,p)$$

Z = normalization constant.

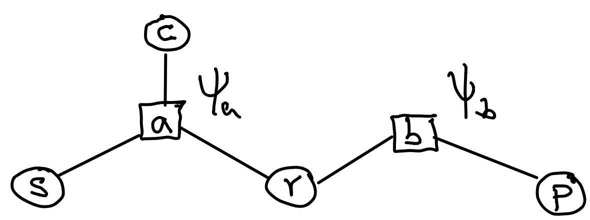
In our example

1111	.1 .3	= .03
1110	.1 .2	= .02
0011	.3 .3	.09
0010	.3 .2	.06
0001	.6 .1	.06
0000	.6 .4	.24
	Z =	.50

Such be 2^5 states

The remaining ones are 0 prob.

View as a graph = factor graph



In general: Factor Graph is a bipartite

The graph will have:

1) variable nodes ; V's
2) factor nodes ; a's $\partial a \equiv$ neig of a
   where $\partial a \equiv$ variables of Ψ_a.

3) for each factor a map:

$$\Psi_a : \{0,1\}^{|\partial a|} \to \mathbb{R}^+$$

<u>Tasks</u>: Given a factor graph G.

1) Compute marginals ie $P(X_i = 1)$
2) Compute conditionals ie $P(X_i = 1 \mid X_j = 0)$
3) Same from dist.
4) Find mode ie $\arg\max_x \{P(x)\}$

<u>Claim</u> For general factor graph computing any of the above is NP-hard.

We will show: 3-SAT $\leq_p$ Marginal
the others are similar.

<u>Recall</u>: 3-CNF is a Boolean formula with 3 literals per clause.

$$\text{3-SAT} \equiv \{ \rho \in \text{3-CNF} \mid \rho \text{ has a satisfying instance} \}$$

eg $F = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_3 \vee \bar{x}_4 \vee \bar{x}_5) \in \text{3-CNF}$

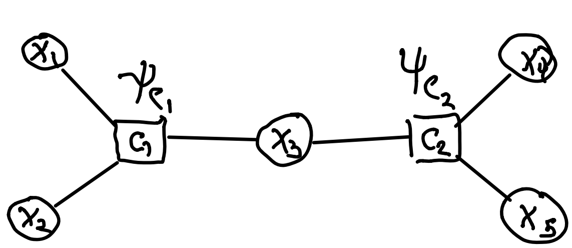
Note $(x_1 \vee \bar{x}_2 \vee x_3)$ is false iff $x_1 = x_3 = F$ & $x_2 = T$

Suppose $\rho \in$ 3-CNF with clauses $C_1, \dots, C_m$.
for each clause c Let ψ_c be the factor s.t.

$$\psi_c \equiv \begin{cases} 1 & \text{if true} \\ 0 & \text{o.w.} \end{cases}$$

In our example:

$$\psi_{C_1}(x_1, x_2, x_3) = \begin{cases} 1 & \text{if true} \\ 0 & \text{o.w} \end{cases}$$



Consider uniform dist over $\{0,1\}^5$

$$P(x_1 \cdots x_5) = \frac{1}{Z} \psi_{C_1}(x_1 x_2 x_3)\, \psi_{C_2}(x_3 x_4 x_5)$$

When $Z \neq 0$ P is uniform dist over satisfying assignments

Thus $P(x_1 = 1) > 0$ iff F is satisfiable.

Thus computing marginals is NP-Hard

<u>Note</u> Computing Z is #P-Complete.

Factor Graphs that are trees

We restrict factor graphs to a tree.

<u>To show</u> For trees factor graph tasks 1), ..., 4) are poly-time.

We will solve the unnormalized prob

$$Z = \sum_{X \in \{0,1\}^n} \prod_{a=1}^{m} \Psi_a(X_{\partial a})$$

$$Z[X_i = b] = \sum_{\substack{X \in \{0,1\}^n \\ X_i = b}} \prod_{a=1}^{m} \Psi_a(X_{\partial a})$$

<u>Note</u>: $P[X_i = b] = Z[X_i = b] / Z$

<u>Goal</u>: Compute $Z[X_i = b]$

We root G at vertex X_i

<u>Subproblems</u>: Rooted subtrees of G.

Let T_w be subtree rooted at w (var or fact)

two types of subtrees.

1) Variable rooted 2) Factor rooted

T_x^a be subtree of T_x with child a

V_w be all variables in T_w (T_w^a)

F_w be all factors in T_w (T_w^a)

$Child(w) \doteq children(w)$

<u>For Variable-rooted</u>:

$\forall i \in [n]$ & $\forall b \in \{0,1\}$

<u>compute</u>: $$V_i(b) = \sum_{\substack{y \in \{0,1\}^{|V_i|} \\ y_i = b}} \prod_{a \in F_i} \psi_a(y_{\partial a})$$

For factor-rooted trees:

$\forall i \in |n|$ & $\forall b \in \{0,1\}$ & $\forall \bar{a} \in \text{child}(x_i)$

Compute:

$$V_i(\bar{a}, b) = \sum_{\substack{y \in \{0,1\}^{|V_{\bar{a}}|} \\ y_i = b}} \psi_{\bar{a}}(y_{\partial \bar{a}}) \prod_{a \in F_i^{\bar{a}}} \psi_a(y_{\partial a})$$

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3) <u>Recurrences</u>:

For variable-rooted subtrees

$$V_i(b) = \begin{cases} \prod_{\bar{a} \in Ch(i)} V_i(\bar{a}, b) & x_i \text{ not leaf} \\ 1 & \text{o.w} \end{cases}$$

For factor-rooted subtrees

$$V_i(a,b) = \begin{cases} \sum\limits_{\substack{y \in \{0,1\}^{\partial a} \\ y_i = b}} \psi_a(y_{\partial a}) \prod\limits_{j \in ch(a)} V_j(y_j) & \text{(not leaf)} \\ \psi_a(b) & \text{if leaf} \end{cases}$$

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4) Correction: Simple induction

Called the "Sum-Product Alg"