# A preface to Geometry

15-750
2/25/19

**Cartesian Coordinates** (René Descates 1637)

d-dimensional space $\equiv$ vectors of d real numbers

the fundamental operation

**Dot Product** $p, q \in \mathbb{R}^d$

$$\langle p, q \rangle = p \cdot q = p^T q = \sum_{i=1}^{d} p_i q_i$$

**Properties**

1) symmetric: $p^T q = q^T p$
2) bilinear: $(\alpha p)^T q = \alpha (p^T q)$
3) **rigid motion invariant**

$M \in \mathbb{R}^{d \times d}$ st $M^T M = I$ then

$$(Mp)^T (Mq) = p^T M^T M q = p^T q$$

4) Euclidean length $\equiv |p| = \sqrt{p^T p}$
5) Cosine $p^T q = |p| \cdot |q| \cdot \cos \theta$
6) Cauchy-Schwartz (1821)

$$(p^T q)^2 \leq (p^T p)(q^T q)$$

# Linear Programming in Fixed Dim

Goal: 1) Intro LP
2) Random Incremental 2D Alg

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**Def** LP $\equiv \max c^T x$
subject $Ax \leq d$ where

$A^{n \times m} \in \mathbb{R}^{n \times m} \wedge x^{m \times 1} \wedge c^{m \times 1} \wedge d^{n \times 1}$

Note $x \leq y$ if $\forall i \; x_i \leq y_i$

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**Def** The LP is feasible if
$\exists x \; Ax \leq d$

**Note** $\{x \mid Ax \leq d\}$ the feasibe region
is convex $\equiv$ closed under convex comb.

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2D case

$$\begin{pmatrix} a_1 & b_1 \\ \vdots & \\ a_n & b_n \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$$

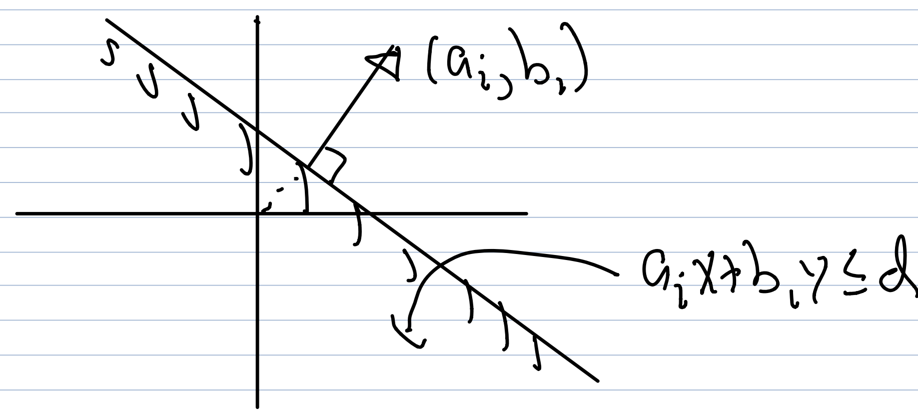
# Geometric View

<u>Note</u> $h_i \equiv \{(x,y) \mid a_i x + b_i y \le d_i\}$

is Half-plane or Half-Space

Consider vector $(a_i, b_i)$

<u>Note</u> $h_i$ is half-plane normal to $(a_i, b_i)$

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<u>Input</u>: to 2D-LP

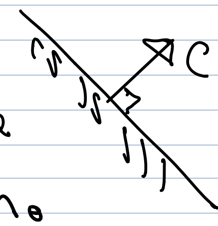
1) Half-planes $\{h_1, \dots, h_n\}$
2) vector $c \in \mathbb{R}^2$

<u>Goal</u>: find $x \in \bigcap_{i=1} h_i$ farthest in $c$ direction.

Def: $\partial h_i \equiv \text{bdry}(h_i)$

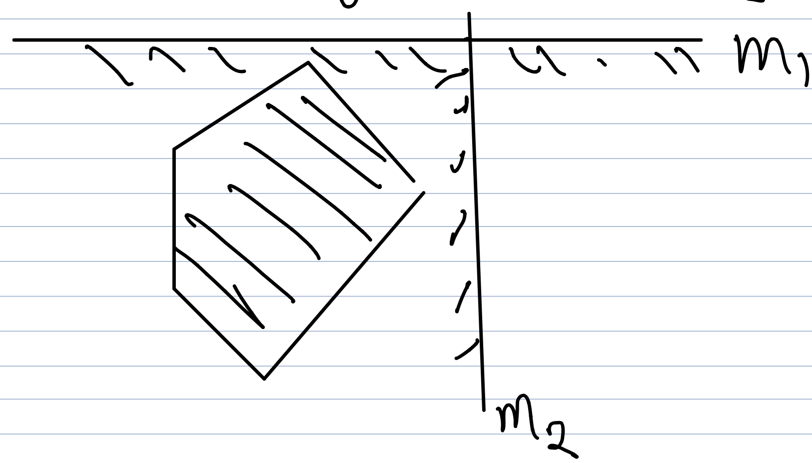
Simplification:

1) No $h_i$ normal to $c$ ie




2) Bounded feasibe region.
3) Given a "bding" box $m_1$ & $m_2$

i.e.

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| $O(n \log n)$ | $O(n)$ |
|---|---|
| Sorting | Selection |
| Convex Hull | LP (fixed dim) |
| Half-Space Inter | ? |
| Meshing | ? |

1D-LP

Input: Constraints $a_i x \le b_i$ $a_i \ne 0$

WLOG $a_i = \pm 1$

2 Types

$C^+ = \{ i \mid x \le b_i \}$

$C^- = \{ i \mid -x \le b_i \}$ ie $-b_i \le x$


$\alpha = \max\{ -b_i \mid i \in C^- \}$

$\beta = \min\{ b_i \mid i \in C^+ \}$

Note: feasible iff $\alpha \le \beta$

If feasible return $= \begin{cases} \beta & \text{if } \operatorname{sign}(c) = 1 \\ \alpha & \text{o.w} \end{cases}$

<u>Thm</u> 1D-LP is $O(n)$ time

# Random Incremental 2D-LP

**Proc:** 2D-LP($m_1, m_2, h_1, \ldots, h_n, c$)

1) $V_0 \leftarrow$ 2D-LP($m_1, m_2, c$)
   ie $V_0 = \partial m_1 \cap \partial m_2$
2) Randomly order $h_1, \ldots, h_n$
3) **for** $i=1$ to $n$ **do**
   **if** $V_{i-1} \in h_i$ **then** $V_i \leftarrow V_{i-1}$
   **else** (Make & solve 1D-LP prob)
      Let $L = \partial h_i$
      $L_1' = L \cap h_1, \ldots L_{i-1}' = L \cap h_{i-1}$
      $c' =$ projection$(c, L)$ note $c' \neq 0$
   $V_i =$ 1D-LP$(L_1', \ldots, L_{i-1}', c')$
   **if** $V_i$ is "undef" report "no solution" & halt.

## Correctness

**Claim:** At time we compute $V_i$

$$V_i = LP(m_1, m_2, \cdots h_i, c)$$

pf Induction on $i$

Base case OK

Suppose $V_{i-1}$ is correct.

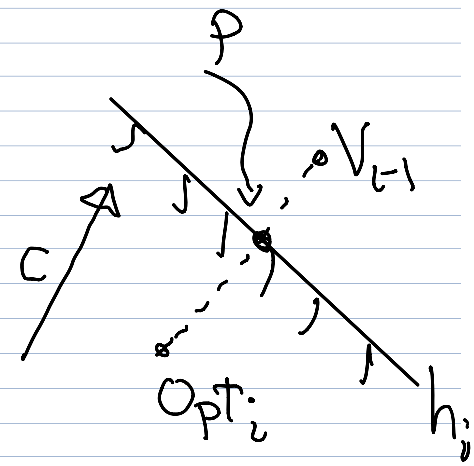
**Case a:** $V_{i-1} \in h_i$ then

$V_{i-1} \in$ feasible region and thus opt.

**Case b:** $V_{i-1} \notin h_i$

**Claim** $V_i \in \partial h_i \equiv L$.

**Picture Proof**




$c^T V_{i-1} \geq c^T Opt_i$

$\Rightarrow$ 1) $c^T p \geq c^T Opt_i$

2) $p$ feasible

$\therefore\ p = Opt_i$

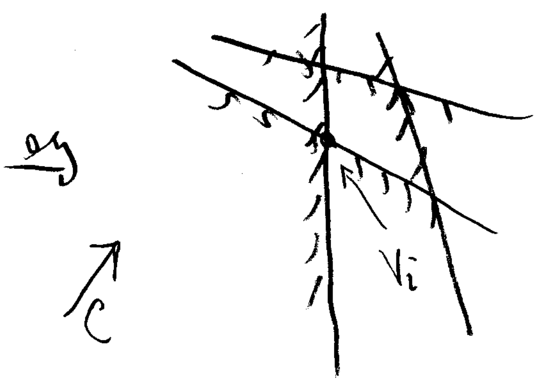
## Timing

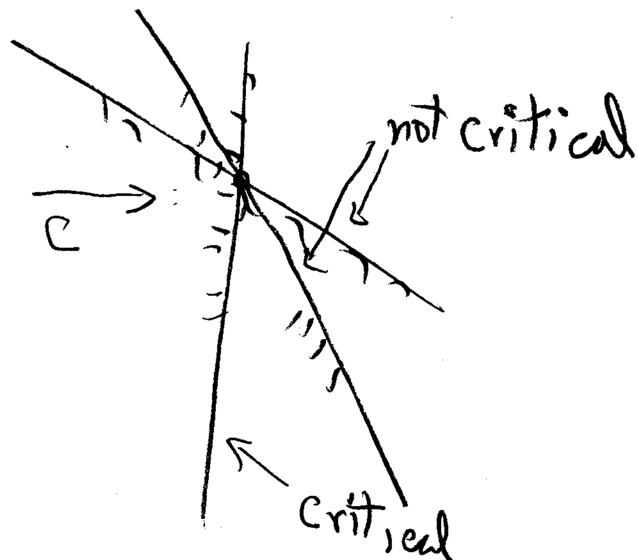
**Claim** 2D-LP is $O(n)$ expected time.

We use backward analysis.

Suppose we remove $h_j$ from $h_1 \cdots h_i$.

**Def** $h_j$ critical if removing it changes opt solution




**Note** At most 2 critical constraints

$$\text{cost}(h_j) = \begin{cases} k & \text{if } h_j \text{ not critical} \\ k \cdot i & \text{o.w.} \end{cases}$$ constant $k$.

Thus worst case for step 3)
is exactly 2 critical constraints.

Let $E_i$ = Expect cost of step 3) at
time $i$

$$E_i \le \left(\left(\frac{2}{i}\right)k \cdot i + \left(\frac{i-2}{i}\right)k\right) \quad \text{some constant } k$$

$$\le 2k + k = 3k$$

Thus total expected work is

$$\sum_{i=1}^{n} 3k = O(n).$$

## Correctness

**Claim:** At time we compute $V_i$

$$V_i = LP(m_1, m_2, \cdots h_i, c)$$

pf Induction on $i$

Base case OK

Suppose $V_{i-1}$ is correct.

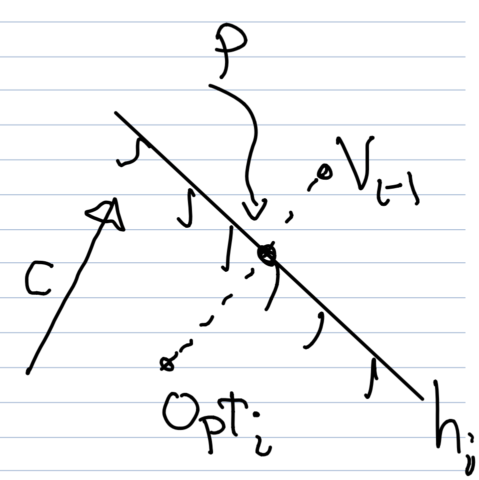
**Case a:** $V_{i-1} \in h_i$ then

$V_{i-1} \in$ feasible region and thus opt.

**Case b:** $V_{i-1} \notin h_i$

**Claim** $V_i \in CH(h_i) \equiv L$.

**Picture Proof**




$c^T V_{i-1} \geq c^T Opt_i$

$\Rightarrow$ 1) $c^T p \geq c^T Opt_i$

2) $p$ feasible

$\therefore p = Opt_i$

# Determining Unbdness or Finding the bding box

LP $\equiv$ Max $C^T X$ subject $AX \leq b$

**Lemma** LP is unbounded iff

1) it is feasible
2) $\exists d$ s.t. $C^T d > 0$ & $Ad \leq 0$

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pf ($\Leftarrow$) By 1) $\exists \bar{x}$ $A\bar{x} \leq b$

pick any $\alpha \geq 0$ & $d$ by 2).

Now: $A(\alpha d + \bar{x}) \leq \alpha A d + A\bar{x} \leq \alpha A d + b \leq b$

thus $\alpha d + \bar{x}$ is feasible $\forall \alpha \geq 0$.

$C^T(\alpha d + \bar{x}) = \alpha C^T d + C^T \bar{x}$

objective goes to $\infty$ with $\alpha$.

($\Rightarrow$) compactness argument

**Lemma** $\exists d$ $Ad < 0$ then $Ax \leq b$ is feasible

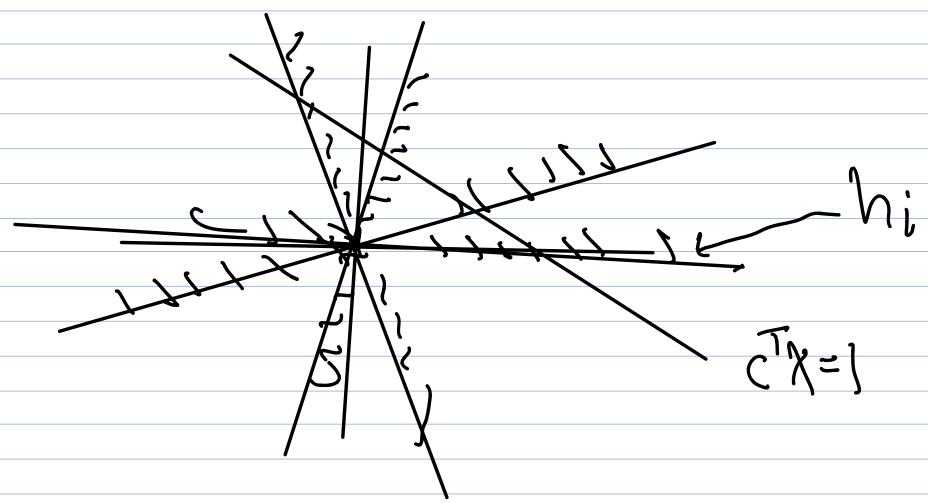
pf $V = Ad$ then $V < 0$ pick $\alpha > 0$ st

$\alpha V \leq b$ thus $A\alpha d \leq b$.

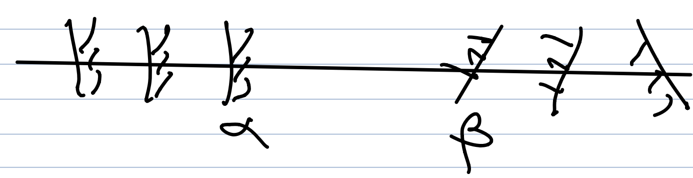
## Finding d for unbdness

d exists iff $\exists d$ s.t. $c^T d = 1$ & $Ad \leq 0$

Our feasible region is $Ax \leq 0$
we intersect it with line $c^T x = 1$



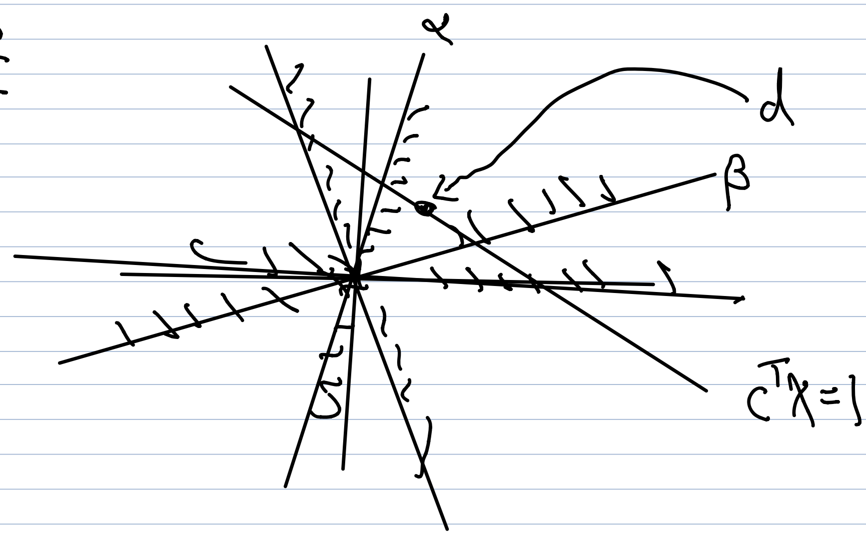

this is a 1D-LP




Note $\exists d$ iff $\beta \geq \alpha$

<u>Claim</u> If $\beta > \alpha$ then $Ax \le b$ is feasible and thus $c^T x$, $Ax \le b$ unbdd.

<u>Pf</u>




Pick $d$ as in figure then $Ad < 0$

Thus $Ax \le b$ is feasible.

There are 3 cases

1) $\alpha < \beta$ 2) $\alpha > \beta$ 3) $\alpha = \beta$

Case 1) $\alpha < \beta$ return "unbded"

Case 2) $\alpha > \beta$ return bounding box:

Assume one halfspace corresponding to $\alpha$
" " $\beta$

Say $h_\alpha$ and $h_\beta$
Return BB $\{h_\alpha, h_\beta\}$

Case 3) $\alpha = \beta$

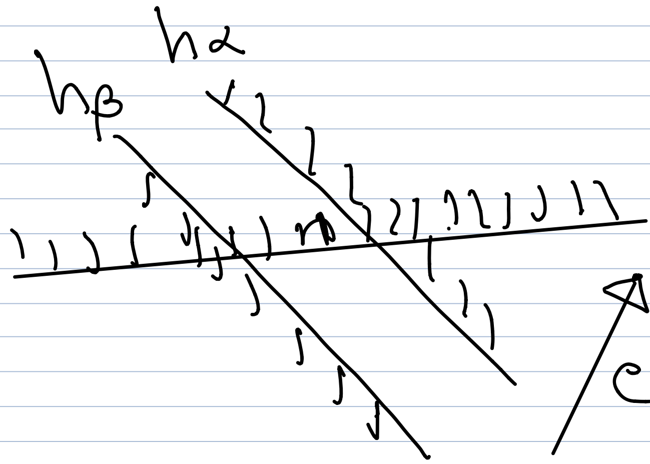
Subcase a $h_\alpha \cap h_\beta = \emptyset$

$Ax \le b$ not feasible

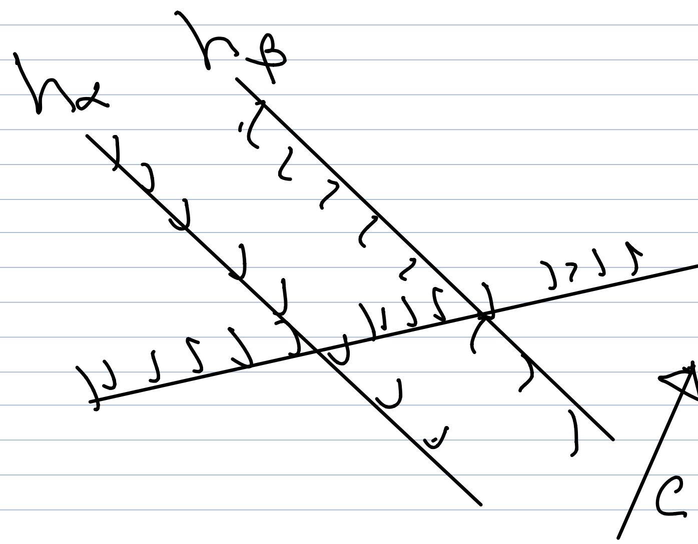
Subcase b $h_\alpha \cap h_\beta \neq \emptyset$

Claim $Ax \le b$ feasible then
$Ax \le b$ $c^T x$ unbded.

Case a




Case b



# LP for fixed dim

**Proc:** d-LP$(h_1, \ldots, h_n, c)$

0) Run Boundedness$(h_1, \ldots, h_n, c)$
   if unbded then quit.
else pick Bounding Box $\{g_1, \ldots, g_d\} \subseteq \{h_1, \ldots, h_n\}$
   Set $V_0 = \bigcap \partial g_i$

1) Let $h_{d+1}, \ldots h_n$ be random order of $H - G$

2) For $i = d+1$ to $n$ do
   if $V_{i-1} \in h_i$ then $V_i \leftarrow V_{i-1}$
   else (Make & Solve (d-1)-LP prob)
   if "undef" report "no solution" & halt
   else $V_i \leftarrow$ (d-1)-LP

<u>Claim</u> d-LP is expected runtime
$C_d \cdot n$ where $C_d = O(2^d \cdot d!)$

We know $C_2 < \infty$

Suppose $C_{d-1} < \infty$ i.e.
$C_{d-1} \cdot n$ bds expected cost to
make & solve $(d-1)$-LP.

<u>Note</u> At most $d$ critical constraints.
Thus by BA we get

$$\mathrm{Cost}(h_i) = \begin{cases} d K_d & \text{if } h_j \text{ not critical} \\ C_{d-1}\, i & \text{o.w.} \end{cases}$$

$$E_i = \frac{\left(d\, C_{d-1} \cdot i + (i-d) K_d\right)}{i} \leq d\, C_{d-1} + d K_d$$

Total Expected $\leq d \cdot n \left(C_{d-1} + K_d\right)$

Assuming $K_d < C_{d-1}$

$C_d \leq d \cdot 2\, C_{d-1}$

$C_d = O(2^d \cdot d!) = O(d!)$