15-750: Graduate Algorithms Mar 31, 2017

Lecture 26: Random Walks on Graphs

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1 Introduction

Definition 1.1. Let $G = (V, E, w)$ be a given (possibly directed graph), denote $w_i = w(v_i) \equiv (\sum_{(i,j)\in E} w_{ij})$, and $p_{ij} \equiv w_{ij}/w_i$. The following process is a **random walk** on G: suppose at a given time we are at $v_i \in V$, we move to v_j with probability p_{ij}.

Example 1.2. $V \equiv$ all orderings or a deck of 52 cards, $p_{ij} \equiv$ probability of going from order i to order j in one shuffle. (Question: Why do professionals play after 5 shuffles? Related to the mixing rate defined below.)

1.1 Two views of a random walk

- Particle view (definition)
- Wave, probability distribution, or large number of simultaneous independent walkers.

  Specifically, let $X^{(i)}$ be the distribution at time i, then $X^{(i+1)} = AD^{-1}X^{(i)}$, where A is the adjacent matrix of G, and D is the diagonal matrix with the degree of each vertex.

1.2 Important Parameters:

- **Access time** (or Hitting time): $H_{ij} \equiv$ Expected time to visit j starting at i
- **Commute time**: $K(i, j) = H(i, j) + H(j, i)$
- **Cover time**: Expected time to visit all nodes, max over all starting nodes
- **Mixing rate**: the time it takes for the distribution induced by a random walk starting at some vertex to converge to the limiting distribution.

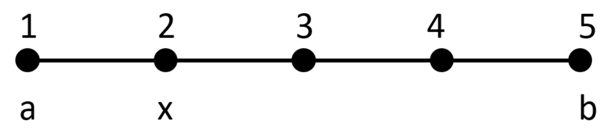
2 Random Walk: the Symmetric Case

- Idea: Do random walk on a network of conductors.
- Input: $G = (V, E, c), c_{ij} = c_{ji}$

Definition 2.1. Consider a random walk starting at x and ending at b, for a given a,

$$h_x = \text{probability we visit } a \text{ before } b$$

Example 2.2. Consider the following graph with unit weight on each edge:



It is straightforward that $h_a = 1$ and $h_b = 0$. What about h_2? An immediate lower bound is $h_2 > 1/2$, since we have a half chance heading towards a in the first move, and some possibility coming back to a from the right hand side. But can we be more precise about h_2?

Claim 2.3. *Suppose $x \neq a, b$, then $h_x = \sum_y p_{xy} h_y$.*

Since $p_{xy} \geq 0$ and $\sum_y p_{xy} = 1$, h_x is a convex combination of its neighbors. In other words, h is harmonic with boundary points a, b.

We can construct an identical electrical problem. Consider $V_a = 1$ and $V_b = 0$, then we have

$$\forall x \neq a, b,\ V_x = \sum_y \frac{c_{xy}}{c_x} V_y.$$

Note that $\frac{c_{xy}}{c_x} = p_{xy}$, by the uniqueness of the solution to the harmonic recursion, we have

$$h = V$$

.

Theorem 2.4. *Set $V_a = 1$, $V_b = 0$ and $x \neq a, b$, then $V_x =$ probability of visiting a before b.*

Back to the example, we can solve the voltage between each two conductors easily, and therefore $h_2 = 3/4$.

3 Interpretation of Current for Random Walk

Consider 1 unit of potential current flow from a to b, say i. What does i_{xy} correspond to in random walk from a to b?

Theorem 3.1. *i_{xy} = Expected net number of traversals of edge e=(x,y) in random walk from a to b.*

4 How to compute hitting time

Definition 4.1. $h(x, b) \equiv$ expected time to reach b from x

$h_x = h(x, b), b$ fixed

Let's write a recurrance: $h_b = 0, x \neq b, h_x = 1 + \Sigma_y h_y P_{xy}$

Let's think of h_x as voltage V_x

$V_b = 0, V_x = 1 + \Sigma_y \frac{c_{xy}}{c_x} V_y$ when $x \neq b$

$c_x V_x = c_x + \Sigma_y c_{xy} V_y$

$c_x V_x - \Sigma_y c_{xy} V_y = c_x$

The left hand side of the above equaiton can be viewed as the vector V dotted with a row of the Laplacian. The right hand side is just the residual current at the corresponding node.

Let $n = b$. Here we have $n-1$ constraints. However, recall that for a connected graph, the Laplacian has rank $n-1$, so the solutions to this system of equations form a 1-dimensional affine subspace. By adding another constraint $V_n = 0$, we would be able to fix a unique solution.

Specifically, define $c = \Sigma_i c_i$, we have

$$LV = \begin{pmatrix} c_1 \\ \vdots \\ c_{n-1} \\ \delta \end{pmatrix}$$

where $\delta = c_n - c$

Algorithm for computing hitting time to V_n

Solve

$$LV = \begin{pmatrix} c_1 \\ \vdots \\ c_{n-1} \\ \delta \end{pmatrix}$$

return V_x

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5 How to compute commute time

Set vertex 1 to be a, vetex n to be b

Solution 5.1. Solve

$$LV^b = \begin{pmatrix} c_1 \\ \vdots \\ c_n - c \end{pmatrix}, LV^a = \begin{pmatrix} c_1 - 1 \\ \vdots \\ c_n \end{pmatrix}$$

$h(1,n) = V_1^b - V_n^b, h(n,1) = V_n^a - V_1^a$

Set $V = V^b - V^a$

$K(1,n) = (V^b - V^a)_1 - (V^b - V^a)_n = V_1 - V_n$

Solution 5.2.

$$L(V^b - V^a) = LV^b - LV^a = \begin{pmatrix} c_1 \\ \vdots \\ c_n - c \end{pmatrix} - \begin{pmatrix} c_1 - c \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} c \\ 0 \\ \vdots \\ 0 \\ -c \end{pmatrix} = c \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}$$

Solve

$$LV = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}$$

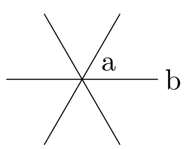
return $c(V_1 - V_n)$

but $(V_1 - V_n) = ER_{1n}$

where ER_{xy} is the effective resistence between two nodes x and y, or how much does voltage drop as current goes through.

Theorem 5.3. $K(a, b) = c \cdot ER_{ab} = 2m \cdot ER_{ab}$

Example 5.4. In this example, we have $n - 1$ nodes strecthing out from one center node a, as depicted by the graph below.



Using the above Theorem, $K(a, b) = 2(n - 1)$