Today

Assume basic NP-Completeness

Let $L$ be a language

$L \in NP$ if $\exists$ short proof of membership

$L \equiv \text{ all 3-colorable graphs,}$

the prove $G$ in 3-color simply exhibit 3-coloring

$L \leq_p L'$ if $\exists x \in L \iff f(x) \in L'$

$a)$ if poly time,

Def: $G$ in graph $H$ is a Hamiltonian Cycle if $H$ is simple cycle containing $V_0.$
Goal: Solve NP-Hard Problems Approximately

We now consider optimization problems.

E.g. 1) Coloring a graph with min # of colors
    a) Finding a min size vertex cover

Let P be an optimization prob.

Def: An alg A is poly-time k-approx alg if:

1) A is poly-time
2) \( A(T) \leq k \cdot Opt_P(T) \)
Vertex-Cover Prob

Input: \( G = (V, E) \)

Output: Minimum size CSV that 'covers' \( E \)

Question: How do we lower \( \text{bd Opt} \)?

One Answer for VC:

Def \( e, e', e \in E \) are independent if \( e \cap e' = \emptyset \)

Let \( E' \subseteq E \) be an independent set of edges.

Claim: \( \text{Opt} \geq |E'| \)

If \( \text{Opt} \) must contain at least one endpoint from each \( e \in E' \).
Algo Approx-VC

1) Find a maximal ind set of edges $E'$
2) Return $V'$ the endpoints of $E'$.

Note  Approx-VC returns a vertex-cover.

Claim  Approx-VC is a poly-time $2$-approx alg.

Proof  1) Clearly Approx-VC is poly-time

2) $|\text{Approx-VC}| = 2|E'| \leq 2|\text{Opt-VC}|$

$(2-\epsilon)$-approx through hard.
Traveling-Salesman Prob

Input: Complete graph \( G=(V,E) \)
Cost func \( C : E \rightarrow \mathbb{R}^+ \)

Output: Min-cost Hamiltonian cycle.

Claim: TSP is NP-Hard

\( \text{Ham-Cycle } \leq^T \text{TSP} \)
\( G=(V,E) \Rightarrow C(u,v) = \begin{cases} 0 & \text{if } (u,v) \in E \\ 1 & \text{otherwise} \end{cases} \)

\( G \in \text{Ham-Cycle } \iff |\text{TSP}(C)| = 0 \)
TSP with Triangle Inequality

**Def** C satisfies the triangle inequality if

\[ \forall u, v, w \in V \quad C(u, w) \leq C(u, v) + C(v, w) \]

**Claim** TSP with tri-inequality is NP-Hard.

**Ref** Ham-Cycle \( \leq_p^T \) TSP with TI

\[ G = (V, E) \quad C((u, v)) = \text{dist}(u, v) \text{ in } G. \]

\[ \text{Opt-TSP}(G, C) = N \text{ iff } G \text{ is Ham-Cycle} \]

**Question:** Lower Bound Opt-TSP?

Let \( T^* \) be Opt Tour.

This \( T^* \) minus some edge is a spanning Tree.

\[ \therefore |\text{MST}| \leq |T^*| \]
**Approx-TSP Alg**

1. Find a MST \( T \) of \( G \)
2. Compute an Euler Tour of \( T \)
3. Remove multiply visited vertices from Euler Tour.

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**EG**

[Diagram of a graph with labeled edges illustrating the Euler tour and Hamiltonian cycle.]
Let $T_{out}$ be an Euler-Tour with extra vertices shunted.

\[ |T_{out}| \leq |ET| \leq 2 \cdot |MST| \leq 2 \cdot Opt-TSP \]

**Thm** If $P \neq NP$ then $\forall \delta > 1 \not\exists$ poly-time $\delta$-approx for TSP.
To Show: \( \text{Ham-Cycle} \leq^T_p \text{\rho-approx TSP} \)

**Input:** \( G = (V,E) \)

**Output:** \( G' = (V, E') \) \( E' = \{(u,v) \mid u \neq v \} \)

\[
C(u,v) = \begin{cases} 
1 & \text{if } (u,v) \in E \\
\rho \cdot n + 1 & \text{otherwise} 
\end{cases} \quad n = |V|
\]

**Note**
- \( G \in \text{Ham-Cycle} \) then \( |\text{TSP}| = n \)
- \( G \in \text{Ham-Cycle} \) then \( |\text{TSP}| \geq (n-1) + \rho n + 1 \)
- \( = (\rho + 1) n \)

Thus, if \( G \in \text{Ham-Cycle} \) & \( \text{\rho-approx TSP} \) returns a \( \rho \)-approx it must be a \( \text{Ham-Cycle} \).
Center Selection Prob

Input: \( P_1, \ldots, P_n \in \mathbb{R}^d \) sites, \( \text{int } K \)
Output: \( C = \{ c_1, \ldots, c_K \} \subset \mathbb{R}^d \) centers

\[ \text{st } \min_{P \in P} \max_{c \in C} \text{dist}(P, C) = r \]

Def: \( \text{dist}(P, C) = \min_{c \in C} \text{dist}(P, c) \)

Note: \( d = 2 \) & \( k = 1 \) we gave a linear time algo.

In general: \( \text{NP-Hard} \)

Goal: \( 2\)-approx, ie \( 2R \)-solution.

2 algorithms

1) \( 2\)-approx given \( R \).
2) \( 2\)-approx without \( R \).
**Simple Greedy** \((P=\{p_1, \ldots, p_n\}, r)\) \(n>k\)

**Init:** \(C = \emptyset\) , \(Q=\{p_1, \ldots, p_n\}\)

**While** \(Q \neq \emptyset\)

1. extract \(p\) from \(Q\) (any \(p\))
2. add \(p\) to \(C\)
3. remove from \(Q\) all \(q\) s.t. \(2r \geq \text{dist}(p, q)\)

**Thm:** Simple Greedy is a 2-Approx (given \(r\))

**Proof:** induction on \(k\)

If \(k=1\) done since \(\text{diam} \leq 2r\)

Assume true for \(k-1\)

Initially: \(P\) has \(k\)-centers of radius \(r\) solution

After one round of while loop

\(P\) will have \((k-1)\)-centers of radius \(r\) solution.

Since we removed a center and points in it.
**Furthest Greedy** $(P, k)$

**Init:** $C = \{p_1\}, P = \{p_2, \ldots, p_n\}, k = 1$

While $k \neq 0$

1) Pick $p \in \{p_2, \ldots, p_n\}$ max dist $(p, C)$
2) Add $p$ to $C$
3) $k = k - 1$

**Thm:** Furthest Greedy is a $2$-Approx

$p^*$
While $\max_{p \in P} \text{dist}(p, C) > 2r$ points

picked could have been pick by simple Greedy

If $\max_{p \in P} \text{dist}(p, C) \leq 2r$ done.