15-750
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Strongly Connected Components

Input: Directed graph $G = (V, E)$

$v, w \in V$

Def $v \equiv w$ if $\exists$ path from v to w and $\exists$ path from w to v

Def R on a set X is an equivalence relation

Reflexive: wRw

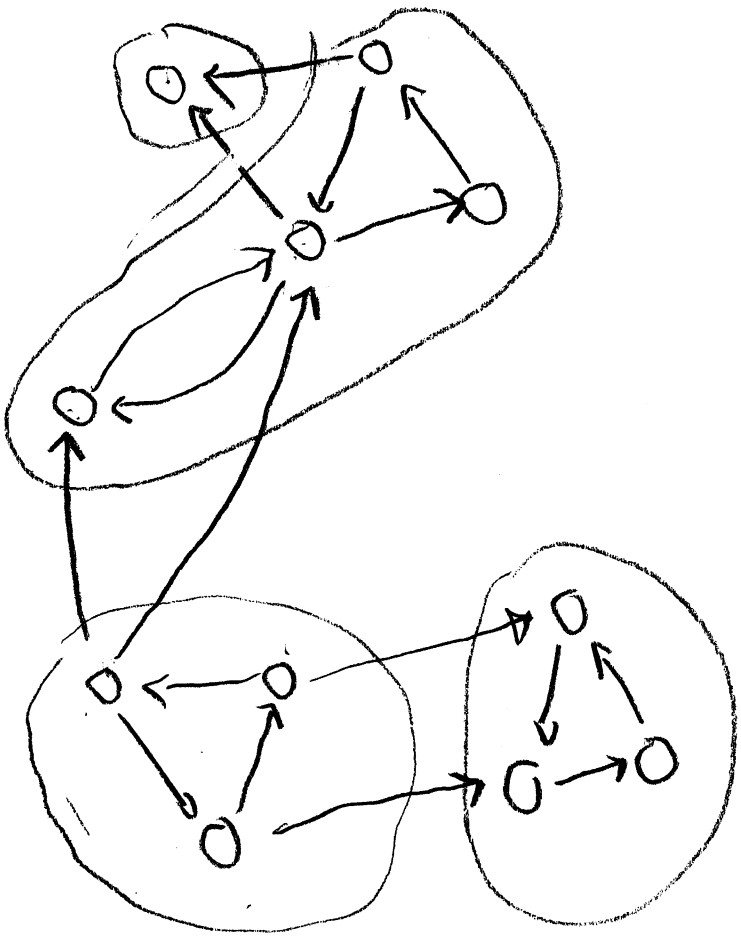
Symmetric: $vRw \Rightarrow wRv$

Transitive: $vRw \wedge wRx \Rightarrow vRx$

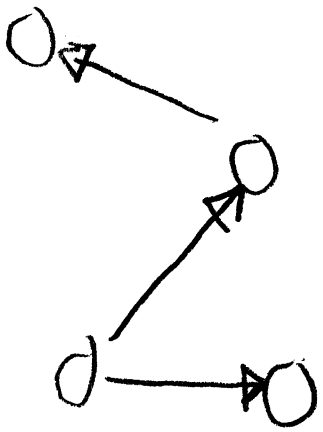
Claim $\equiv$ is an equivalence relation on V

Def The equivalence class are the strongly connected components or strong components

An Example:



Condensation



Alg: DFS(G)

1) $\forall u \in V$ color(u) ← white ; time ← 0

2) $\forall u \in V$ if color(u) ≡ white then DFS-visit(u)

what order?

DFS-visit(u)

1) color(u) ← gray ; push-time(u) ← time++

2) $\forall v \in Adj(u)$

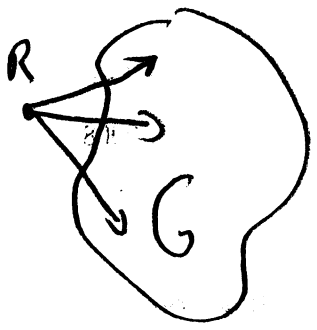
if color(v) ≡ white then DFS-Visit(v)

3) color(u) ← black ; pop-time(u) ← time++

note: dfs(u) ≡ push-time(u)

Conceptually Simplify DFS

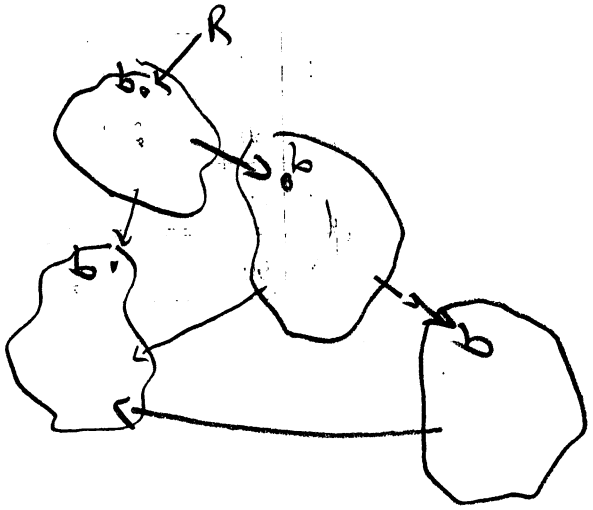
Add a super vertex R

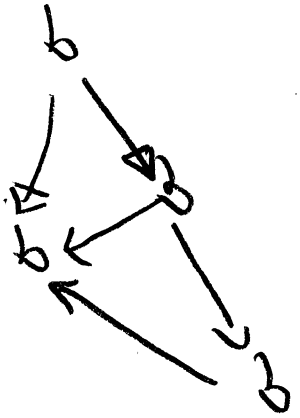


1) Add new vertex R
2) Add new edges from R to all old vertices
3) run DFS(R)

Fix the DFS

Def For each SC the first vertex visited $b \in SC$ is called the **base** node of SC.





1) In our case, DFS Forest is a Tree rooted at R.
   T is an ordered Tree by push-time.
2) We can ignore forward edges! Why?

3) All cross edges go from right to left.

Def Let T be our DFS Tree
$T_v \equiv$ subtree rooted at v.

Lemma 1 If $T_v \cap T_w = \emptyset$ then
pushtimes(T_v) < pushtimes(T_w) or
pushtimes(T_w) < pushtimes(T_v)

Pf Note: pushtimes are just preorder times for T.

Lemma 2 If b is the base for X, a SC, then

1) If P is a path from b to $v \in X$ & $w \in P$ then $w \in X$

2) If $v \in X$ then v is a descendant of b in T.

Pf

Claim 1 Suppose $w \in P$, a path from b to v.

a) bPw is a path from b to w

b) the path from w to b will consist of
   i) wPv in P.
   ii) path from v to b (by hypothesis).

Pf of claim 2

Will show: P is a path from b to v then

$V(P) \subseteq T_b$

Pf induction $|P|$

i) $|P|=1$ then done

Consider first edge $e=(x,y)$ of P st $y \notin T_b$

Then e is either a cross edge

or a back edge

In either case $push(y) < push(b)$

A contra!

Finding the Base Nodes

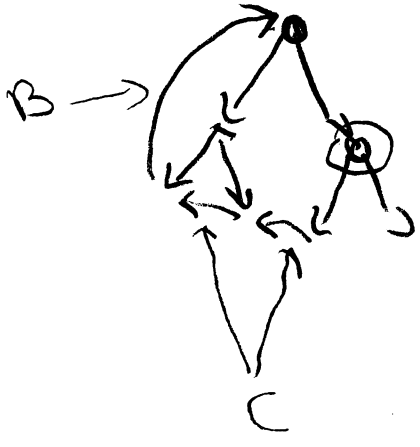
Note: SC are just the forest of T obtained by removing the edge into each base node.

Idea: Use something like lowpoint from Biconnected components

Prob: Tree, Back, Forward edge are OK

Cross edges are a problem.

eg need them



They can give wrong answer

push(a)
push(b)

Solution: Determine when a Cross edge Leaves the SC.

Remove each SC when base is popped.

Main Tool:

$u \in X \subseteq V$ X is SC.

$$\text{lowlink}(u) = \min\{\text{push}(v) \mid v \in \text{Path of form } T^*[C,B]^+ \ \& \ v \in X\}$$

Claim: $\text{lowlink}(u) \geq \text{push}(u)$ iff u is a base node.

Component(G)

1) Init(S: Stack), time=0, $\forall v$ color(v) ← white

2) If color(u) = white then Strong(u)

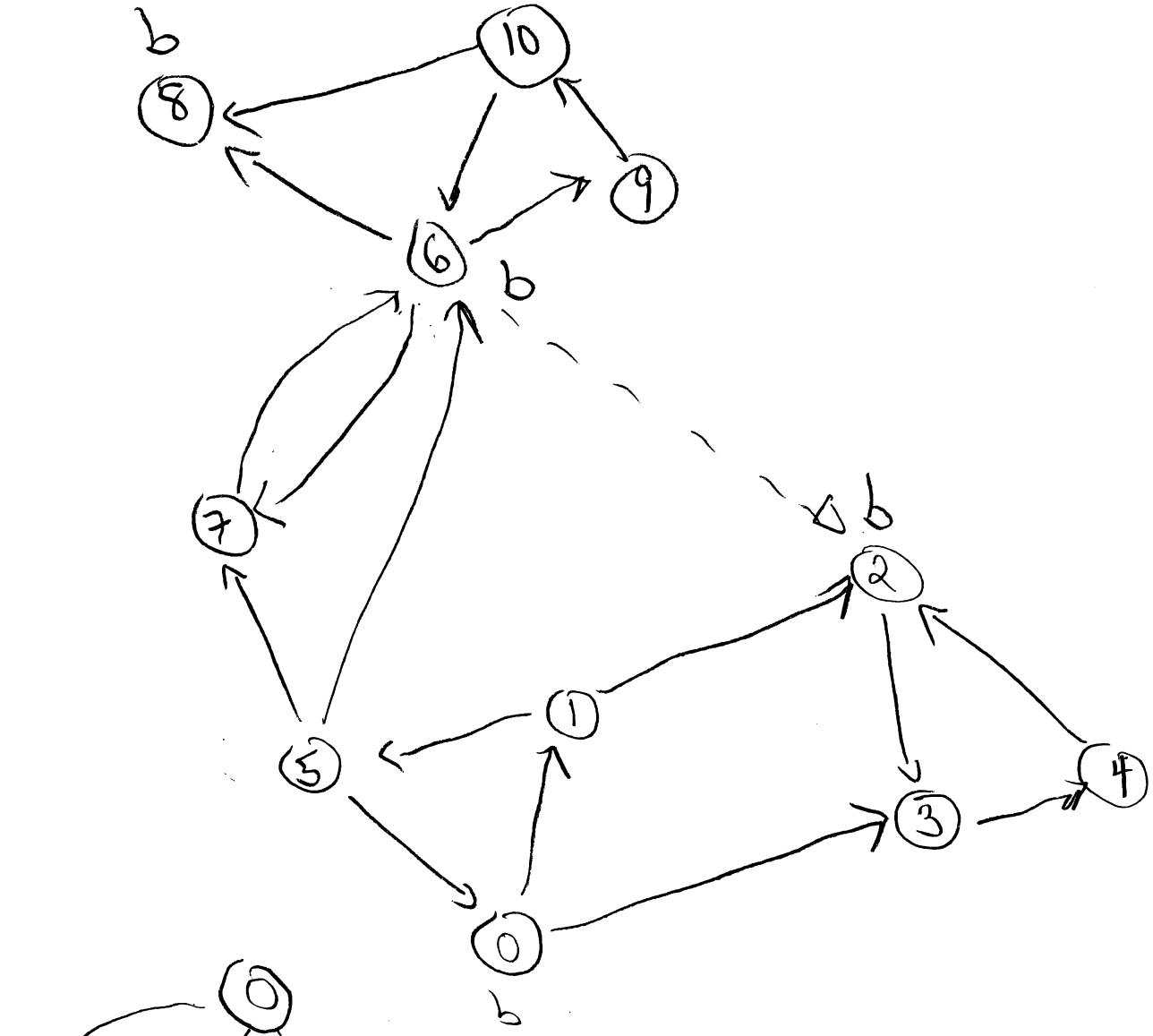
Strong(u)

1) color(u) ← gray ; dfs(u) ← lowlink(u) ← time++
   push(u, S)

2) $\forall v \in Adj(u)$
   if color(v) = white then
      strong(v) ; ll(u) = min{ll(u), ll(v)}
   else if [v ∈ S]
      then ll(u) ← Min{ll(u), dfs(v)}

3) if ll(u) = dfs(u) [u is a base vertex]
   then while [top(s) ≠ u] pop(s) ; pop(s)

end Strong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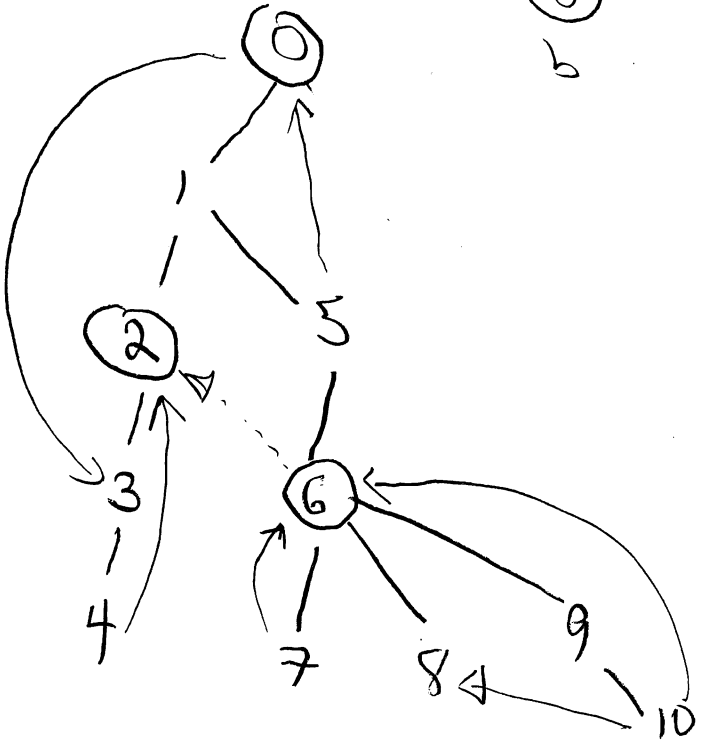
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