15-750
2/12/16

# Graph Spanners

1) $G=(V,E)$ undirected unweighted
2) $H \subseteq G$ is a $K$-spanner if
   $\forall x,y \in V \quad dist_H(x,y) \leq K \cdot dist_G(x,y)$
3) $K$ is stretch factor
4) Goal: $\text{Min} |E_H|$

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Known:

**Thm** $\exists (2K-1)$-spanner with $\frac{1}{2}(n^{1+1/K})$ edges.

Girth: min size cycle in $G$.

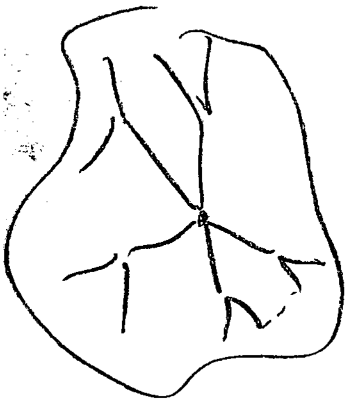
**Conjecture** (Erdos) $\exists G$ s.t.
1) $|E| \doteq \Omega(n^{1+1/K})$
2) $Girth(G) = 2K+1$

Today

<u>Thm</u> 4K+1 Spanner with $O(n^{1+1/k})$ edges
4K+1 expected stretch.

Construction
1) Set $\beta = \log n / 2k$
2) Compute a low dia decomp.
3) Include the spanning forest.
4) For each boundary vertex $v$
add <u>one</u> edge to each adj cluster.

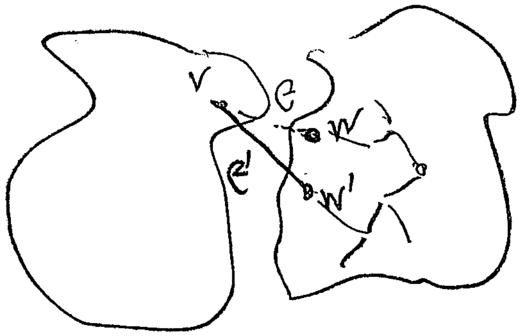
<u>Stretch</u>
a) edge e is internal to cluster



$str(e) \leq dia \leq MaxDia$

$E(MaxDia) \leq \frac{2 \ln n}{\beta} = 4k$

b) edge e is between C & C'




$\exists e' = (v, w')$

$str(e) \le dia + 1 \le MaxDia + 1$

$E(str(e)) \le 2K + 1$

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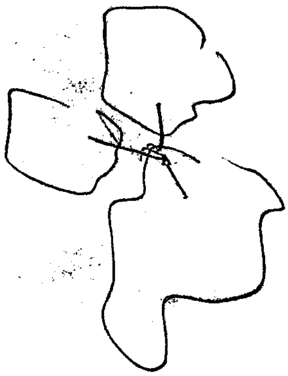
Size of H



1) $e \in F \subseteq H$ at most $n-1$

2) e is intercluster

$\# \le n \cdot Ex[degree_H(x)]$

intercluster degree $(x)$

$\le$ # clusters intersecting $(Neig(x))$

Let $B \subseteq G$ and $dia(B) = d$

Random Variable $Cluster(B) = D$

$$= \left| \{ C \in Cluster \mid C \cap B \neq \emptyset \} \right|$$

<u>Claim</u> $E[Cluster(B)] \leq e^{d \cdot \beta}$

<u>pf</u> $X_1, \ldots, X_t$ iid $Exp(\beta)$

$$Prob[\delta \leq X_1, \ldots, X_t \leq d+\delta \mid \delta \leq X_1, \ldots, X_t]$$

$$= Prob[X_1, \ldots, X_t \leq d] = Prob[X_1 \leq d]^t$$

$$= (1 - e^{-d\beta})^t$$

$$Ex[D] = \sum_{t=1}^{\infty} Prob[D \geq t]$$

$$= \sum_{t=1}^{\infty} (1 - e^{-d\beta})^{t-1} \qquad 1 + \alpha + \alpha^2 \cdots = \frac{1}{1-\alpha}$$

$$= \frac{1}{1-(1-e^{-d\beta})} = \frac{1}{e^{-d\beta}} = e^{d\beta} \qquad *$$

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Recall $\beta = \frac{\log n}{2k}$ $\quad d = 2$

$$* \quad e^{d \cdot \beta} = e^{\frac{\log n}{k}} = n^{1/k}$$

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Exp # of inter cluster

$$n \cdot n^{1/k} = n^{1 + 1/k}$$