

15-750
4/18/16

Yet Another way to Search a Graph

So far we have used

1) BFS

2) DFS

3) Today: Random Walk!

In general random walks maybe slow to search a graph.

We will use tools from to analyze & get fast algorithms

1) EE resistance, energy

2) Linear algebra

1) linear systems

a) Eigenvalues & vectors

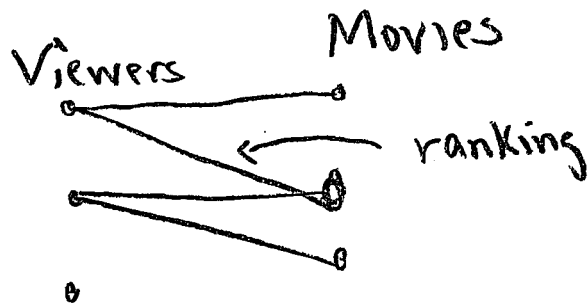
3) Matrix Chernoff bounds

4) Numerical Analysis.

Resistive Model of a Graph & Random Walks

15-8 JAN
9/9/13

Motivation: Making a recommendation
(NETFLIX)



Question: Should we recommend M to V ?
 $\text{Score}(V, M)$

Idea 1 $\text{Score}(V, M) = \frac{1}{\text{graph dist from } V \text{ to } M}$

$$W_{ij} = \frac{1}{\text{rank}_{ij}}$$

$$\text{Score}(V, M) = \frac{1}{\min_{VPM} W(P)}$$

Idea 2 $W(P) = \min_{e \in P} (\text{rank}(e))$

$$\text{Score}(V, M) = \max_{VPM} W(P)$$

Problem For 1) and 2) extra paths do not improve score

Idea 3 $\text{Score}(V, M) \equiv \text{Max flow from } V \text{ to } M.$

Problem Shorter paths do not improve score

Idea 4 View edges as conductors

$\text{Score}(V, M) = \text{effective conductance}$

Idea 5 Consider random walk from V to M

$\text{Score}(V, M) = \text{hit}(V, M) + \text{hit}(M, V)$

$\text{hit}(V, M) \equiv \text{Expect length random walk from } V, M$

We show 4) & 5) are related.

Hypothesis:

effective conductance &

Commute time $\equiv \text{hit}(v, m) + \text{hit}(m, v)$

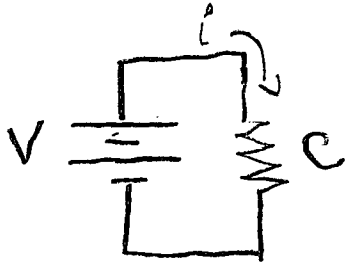
are better scores.

To do:

- 1) Give formal definitions.
- 2) Develop basic theory.
- 3) Give efficient algs.
- 4) Find apps.

Resistance Theory

Ohms Law: $C \equiv$ conductance



$R \equiv$ resistance

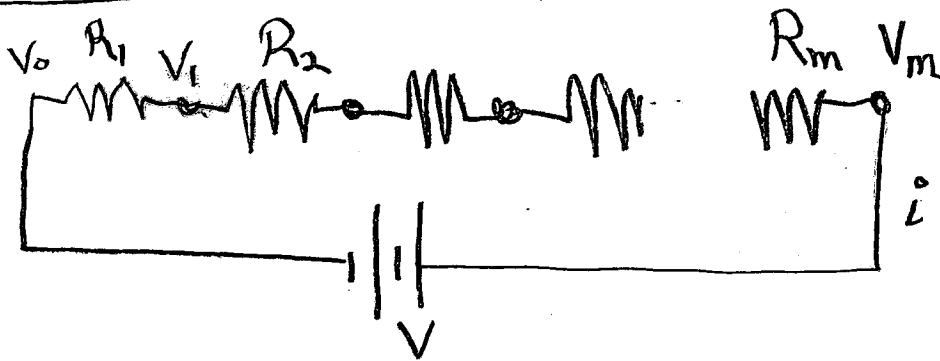
$V \equiv$ voltage

$i \equiv$ current

$$C = 1/R \quad i = C \cdot V = V/R$$

Facts HW

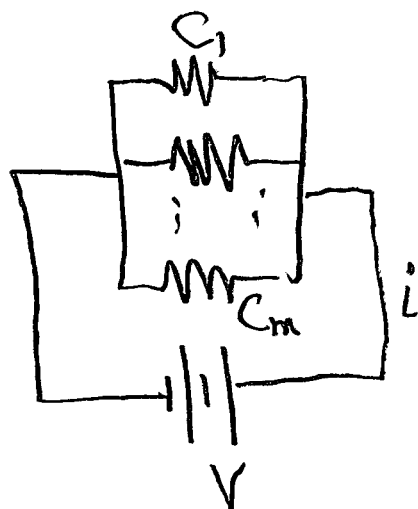
Resistors in series



$$R = R_1 + \dots + R_m \quad C = \frac{1}{(1/C_1 + \dots + 1/C_m)} = ?$$

ie. $i = V/R$

Conductors in Parallel

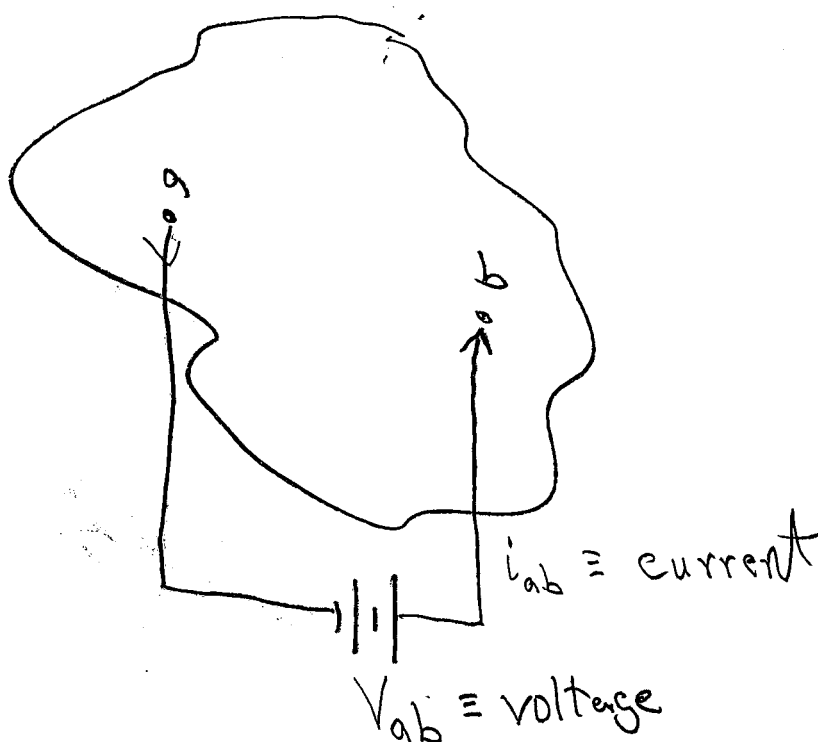


$$C = C_1 + \dots + C_m$$

$$i = V \cdot C$$

Effective Resistance/Conductance

Let G be a network of resistors

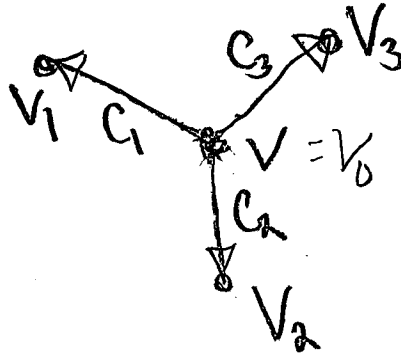


Def $R_{ab} = V_{ab}/i_{ab}$ $C_{ab} = 1/R_{ab}$

Computing effective resistance

Case: Kirchhoff's Law is $\text{flow in} = \text{flow out}$

an example



by Ohm's Law

$$i_1 = C_1(V - V_1)$$

$$i_2 = C_2(V - V_2)$$

$$i_3 = C_3(V - V_3)$$

$$\text{residual current} \equiv i_1 + i_2 + i_3$$

by Kirchhoff

$$i_1 + i_2 + i_3 = 0$$

$$C_1(V - V_1) + C_2(V - V_2) + C_3(V - V_3) = 0$$

$$(C_1 + C_2 + C_3)V = C_1V_1 + C_2V_2 + C_3V_3$$

$$C = C_1 + C_2 + C_3$$

$$CV = C_1 V_1 + C_2 V_2 + C_3 V_3$$

$$V = \frac{C_1}{C} V_1 + \frac{C_2}{C} V_2 + \frac{C_3}{C} V_3$$

V is convex combination of V_1, V_2, V_3

$$\text{residual current} = CV - C_1 V_1 - C_2 V_2 - C_3 V_3$$

The general case:

$$G = (V, E, C) \quad C: E \rightarrow \mathbb{R}^+$$

$$V = \{V_1, \dots, V_n\}$$

$$d(V_i) = \sum_{(i,j) \in E} C_{ij}$$

Def: $A_{ij} = \begin{cases} C_{ij} & \text{if } (i,j) \in E \\ 0 & \text{o.w} \end{cases}$

$$\text{Laplacian}(G) = L(G) = L$$

$$L_{ij} = \begin{cases} d(v_i) & \text{if } i=j \\ -C_{ij} & \text{if } (i,j) \in E \\ 0 & \text{o.w} \end{cases}$$

ie $L = D - A$ where $D = \begin{pmatrix} d(v_1) & & 0 \\ & \ddots & \\ 0 & & d(v_n) \end{pmatrix}$

Let V be a voltage setting of nodes

Note $(LV)_i = \text{residual current at } V_i$

Inverse: We inject currents and get voltages.

The net injected must be zero!

Goal: R_{in}

method 1 solve
$$h \begin{pmatrix} 0 \\ V_1 \\ \vdots \\ V_{n-1} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -i \end{pmatrix} (*)$$

$$\tilde{V} = V/R$$

$$\tilde{V} = V_{in} - V$$

$$R = 1/i$$

return $1/i$

$(*)$ is called a boundary valued prob.

In our case V_1 & V_n are the bdary

$(V_1, \dots, V_n)$ is called harmonic

because $V_i \in \text{interior} \Rightarrow$

V_i is convex combination of neighbors

Maximum Principle If f is harmonic
then min & max are on bdary

Pt $V \in \text{interior}$ then $\exists$ neig V_i & V_j st
 $V_i \leq V \leq V_j$

Uniqueness Principle If f & g are harmonic
with same bdary values then $f = g$

Pt $f - g$ is harmonic with zero on bdary

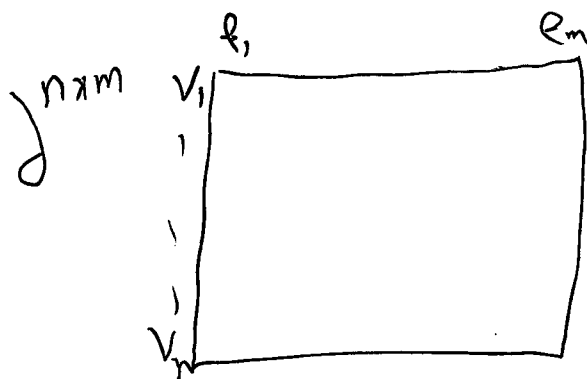
$$\Rightarrow f - g \equiv 0 \Rightarrow f = g$$

Method 2 solve $LV = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}$ return $R_m = V_1 - V_n$

Does V exist?

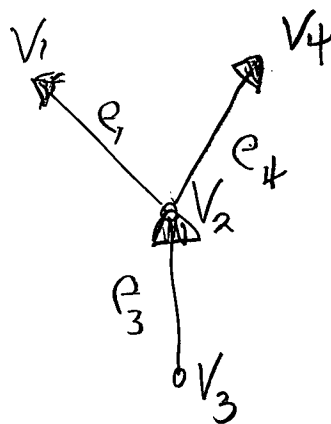
Another way to view the Laplacian

Boundary Operator (Vertex-Edge Matrix)



Orient each edge.

	e_1	e_2	e_3
v_1	1	0	0
v_2	-1	-1	1
v_3	0	0	-1
v_4	0	1	0



Let $C_1, \dots, C_m \equiv$ conductance of $e_1, \dots, e_m$

$$C = \begin{pmatrix} C_1 & & 0 \\ & \ddots & \\ 0 & & C_m \end{pmatrix}$$

Note $\partial^T V \equiv$ voltage drop across each edge
 $C \partial^T V \equiv$ current flow "

$\partial C \partial^T V \equiv$ residual current at each vertex

Thus $L = \partial C \partial^T$

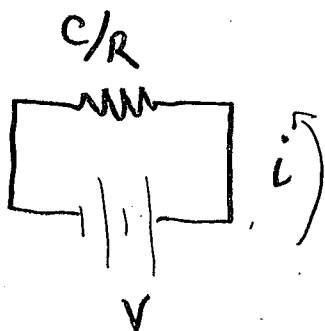
Assume G is a connected graph

$$x^T \partial C \partial^T x = (\partial x)^T C \partial x = \sum C_{ij} (x_i - x_j)^2 \geq 0$$

$$x^T L x = 0 \text{ iff } \forall e_{ij} \quad (x_i - x_j)^2 = 0 = (x_i - x_j)$$

$$\text{Ker}(L) = \langle \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \rangle \quad \text{rank}(L) = n-1$$

Current & Energy/Power Dissipation



Newton Energy $\equiv$ Force $\cdot$ speed
 $\equiv$ Volt $\cdot$ current

$$i = CV$$

$$iR = V$$

$$\begin{aligned} &\equiv V \cdot i \\ &\equiv CV^2 \\ &\equiv i^2 R \end{aligned}$$

Network

$$E = \frac{1}{2} \sum_{x,y} i_{xy} (V_x - V_y)$$

$$V^T L V = V^T \partial C \partial^T V = (\partial^T V)^T C (\partial^T V)$$

$$= \sum_{\substack{\text{oriented} \\ (x,y) \in E}} C_{xy} (V_x - V_y)^2 = E$$

Two Types of Flow

Def: A flow $f: E \rightarrow \mathbb{R}$ (oriented edges)

Def A potential flows $= \{ C \partial^T V \mid V \in \mathbb{R}^n \} = P_G$

Def A circulations (flow) $= \{ f \in \mathbb{R}^m \mid \partial f = 0 \} = C_G$

Assume G is connected & T spanning tree

Claim C_G is a subspace & $\dim(C_G) = m - n + 1$

$m = \# \text{ edges}$

1) Subspace easy

2) $E \setminus T \equiv$ non tree edges of G

$$|E \setminus T| = m - n + 1$$

3) Claim any flow on $E \setminus T$ can be extended to C_G on G . (HW)

4) $f, g \in C_G$ and $f \setminus T = g \setminus T$ then $f = g$.

Claim $f_c \in C_G$ & $g_p \in P_G$ then $f_c^T R g_p = 0$
 when $R = \begin{pmatrix} R_1 & 0 \\ 0 & R_m \end{pmatrix}$

Pf $\exists v$ st $g_p = C \partial v$,

$$f_c^T R g_p = f_c^T R C \partial v = f_c^T \partial v = (\partial f_c)^T v = 0^T v = 0$$

note $RC = I$

Thus C_G, P_G spans $\mathbb{R}^m$ (all flows)

ie $\forall f \in \mathbb{R}^m \exists ! f_c \& f_p$ st $f = f_c + f_p$

Def $f_a = \sum_{a \neq b} f_{ab}$

Def f is a unit-flow from a to b if:

1) f is a flow

2) $-f_a = f_b = 1$

3) $f_x = 0$ for $x \neq a, b$.

Thomson's Principle

- 1) f is unit Potential flow from a to b
- 2) g is any flow from a to b .

then $f^T R f \leq g^T R g$

pf

We know that $g = f + f_c$ $f_c \equiv$ circulation

$$\begin{aligned} g^T R g &= (f + f_c)^T R (f + f_c) = f^T R f + \cancel{2 f_c^T R f} + f_c^T R f_c \\ &= f^T R f + f_c^T R f_c \geq f^T R f \end{aligned}$$

note

Def or Thm Effective resistance from a to b

$$ER_{ab} = f_p^T R f_p$$

$f_p \equiv$ unit potential flow from a to b .

Rayleigh's Monotonicity Law

$$\bar{R} \geq R \text{ then } \bar{ER}_{ab} \geq ER_{ab}$$

pf $f \equiv$ unit potential flow in G_R
 $g \equiv$ in $G_{\bar{R}}$

$$\bar{ER}_{ab} = g^T \bar{R} g = \sum_{e \in G} g_e^2 \bar{R}_e$$

$$\geq \sum_{e \in G} g_e^2 R_e$$

$$\geq \sum_{e \in G} f_e^2 R_e \quad (\text{Thomson})$$

$$= f_e^T R f_e = ER_{ab}$$

HW) Show that R_{ab} is a metric space

ie 1) $R_{ab} \geq 0$

2) $R_{ab} = 0$ iff $a = b$

3) $R_{ab} = R_{ba}$

4) $R_{ac} \leq R_{ab} + R_{bc}$