15-750
4/6/16

Parallel Maximal Independent Set

Luby's Algorithm

Input: Graph $G=(V,E)$ (undirected)

Def $I \subseteq V$ is *independent* if no pair share an edge.

Indep set I is

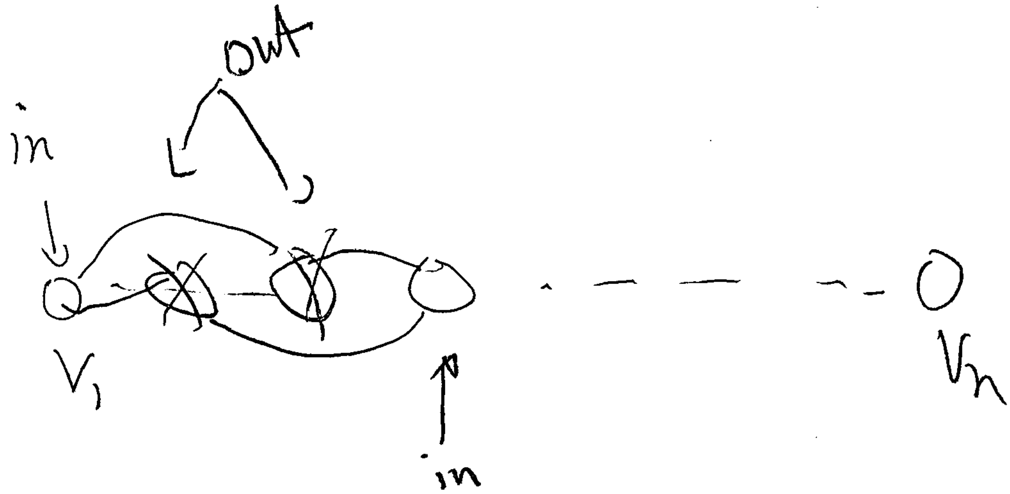
maximal if $\forall w \in V-I$ $w \cup I$ is not ind

maximum if $\forall$ ind sets I' $|I| \geq |I'|$

Note: Finding a maximum ind set is NP-Hard

The simple greedy alg computes a maximal ind in $O(m+n)$ time.

<u>Note</u> Computing this ind set is P-Complete (The lexiographically first MIS)



<u>Note</u> Our List-Rank & Parallel-Tree alg were of following form

1) Find a large independent set of nodes.
2) Process them (remove them)

<u>Def</u> MIS ≡ Maximal Ind Set of G.

Goal: Parallel MIS Alg!

We consider Alg of form for $G=(V,E)$

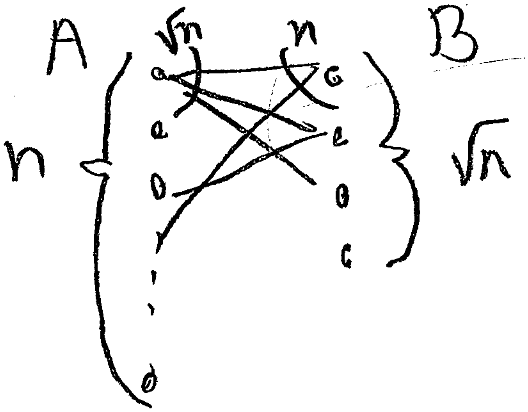
1) Each node v tries to join with prob(v).
2) Break ties giving set $I \subseteq V$
3) Remove I & Neig(I) and all adj edges.
4) Repeat

Suppose $\text{Prob}(v) \approx \frac{1}{d_v}$ $\quad d_v \equiv$ degree of v.

<u>The hope</u> After each round $|V'| \le c|V|$ $c<1$

Bad Example:

Input: $K_{n,\sqrt{n}}$



$$\text{Expect}(|I \cap A|) \approx n\left(\frac{1}{\sqrt{n}}\right) \approx \sqrt{n}$$

$$\text{Expect}(|I \cap B|) \approx \sqrt{n}\left(\frac{1}{n}\right) = \frac{1}{\sqrt{n}}$$

$$|I \cap B| \approx 0$$

Expect $|V'| \approx |V|$

But $E' \not\approx \emptyset$

Randomized Parallel MIS

$MIS(G=(V,E))$

0) Add singeltons to I
1) Each v picks random $p(v) \in [0,1]$
2) V joins I if $p(v) < p(Neigh(v))$
3) $V' = V \setminus I \cup Neigh(I)$

   $E' = E \setminus Edges(I \cup Neigh(I))$
4) return $I \cup MIS(G'=(V',E'))$

Note v joins with prob $= \frac{1}{d_v+1}$

Let $Y = \#$ edges removed (Random Variable)

$$Y_{uv} = \begin{cases} 1 & \text{if } (uv) \text{ is removed} \\ 0 & \text{o.w.} \end{cases} \qquad (u,v) \in E$$

Thus $2Y = \sum_{(uv) \in E} Y_{uv}$

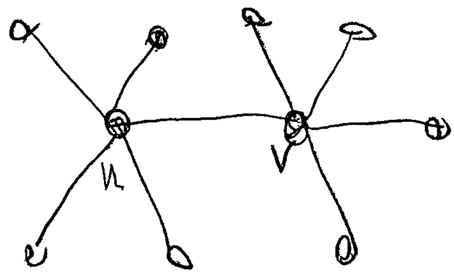
<u>Thm</u> $2\,Exp(Y) \geq |E|$

ie Expected number of edge removed $\geq \frac{|E|}{2}$.

Consider: for $(uv) \in E$

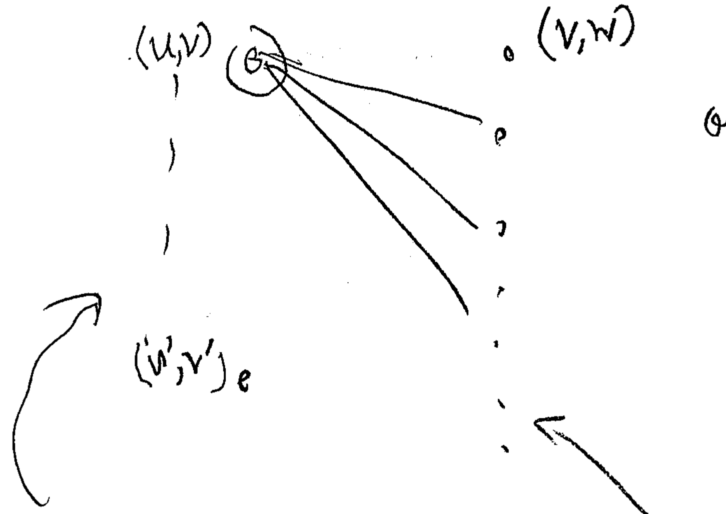
$$X_{uv} = \begin{cases} 1 & \text{if } f(u) < f(Neig(uv) \setminus u) \\ 0 & \text{o.w.} \end{cases}$$

<u>Claim</u> $Exp(X_{uv}) = Prob[X_{uv} = 1] \geq \frac{1}{d_u + d_v}$ (u & v may share nodes)



Claim $\sum_{uv \in E} d_v X_{uv} \le 2Y$

Consider graph on edges



(uv, vw) if $u \neq w$
or (uv, uv)

out degree $\equiv d_v$

indegree $\equiv d_v$

For given (v,w) at most one (u,v) st $X_{uv} = 1$

$$\mathrm{Exp}\left(\sum_{uv \in E} d_v X_{uv} + d_u X_{vu} \right)$$

$$= \sum d_v \,\mathrm{Exp}(X_{uv}) + d_u \,\mathrm{Exp}(X_{vu})$$

$$\geq \sum \frac{d_v}{d_u + d_v} + \frac{d_u}{d_v + d_u} = \sum_{uv \in E} 1 = |E| \leq 2\,\mathrm{Exp}(Y)$$

<u>Thm</u> Random Parallel MIS terminates in $O(\lg n)$ rounds with high prob.

Pf We use Chernoff bds.

Consider random variable

$$Z_i = \begin{cases} 1 \text{ if round } i \text{ deletes at least } 1/3 \text{ remaining edges.} \\ 0 \text{ o.w.} \end{cases}$$

Markov $\text{Prob}[Z_i = 1] \geq 2/3$

<u>note</u> Z_i ind Z_j $i \neq j$

Thus using Chernoff we get Thm