The Max Flow Prob

2/29/16
15-750

Input: 1) Flow network $G(V,E)$ directed (oriented)

2) Edge capacities $c: V \times V \to \mathbb{R}$

$c(u,v) \geq 0$

eg $(u,v) \notin E$ then $c(u,v) = 0$

3) $s, t \in V$ $s \equiv$ source & $t \equiv$ sink

Goal: A flow $f: V \times V \to \mathbb{R}$ s.t.

1) Capacity constraints $f(u,v) \leq c(u,v)$

2) Skewed symmetric $f(u,v) = -f(v,u)$

3) Flow at a vertex $u \in V - \{s,t\}$

$$\sum_{v \in V} f(u,v) = 0$$ ie flow-in = flow-out

Net flow $\equiv |f| = \sum_{v \in V} f(s,v)$

Max-Flow Prob:

input: Flow-Network $G=(V,E),s,t,c$

output: Flow f with max net-flow

Do examples at end of lecture.

Residual Networks

Network $G=(V,E)$ & flow f

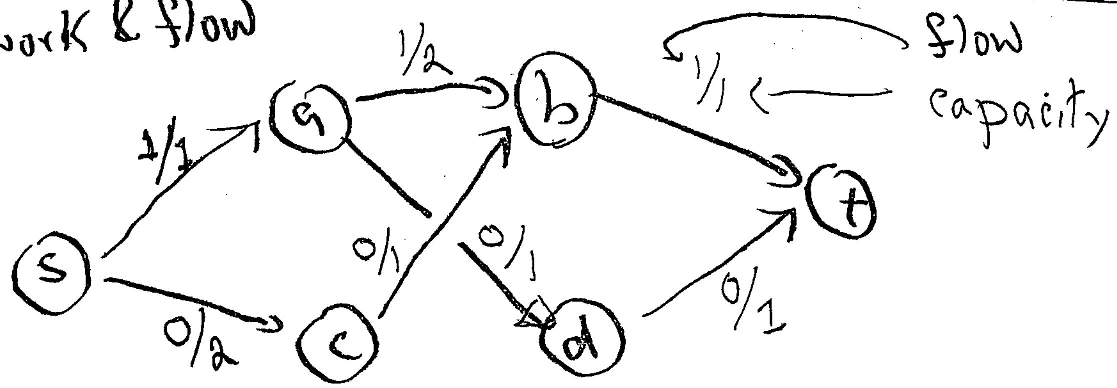
Residual Capacity: $c_f(u,v) = c(u,v) - f(u,v)$

Residual Network:

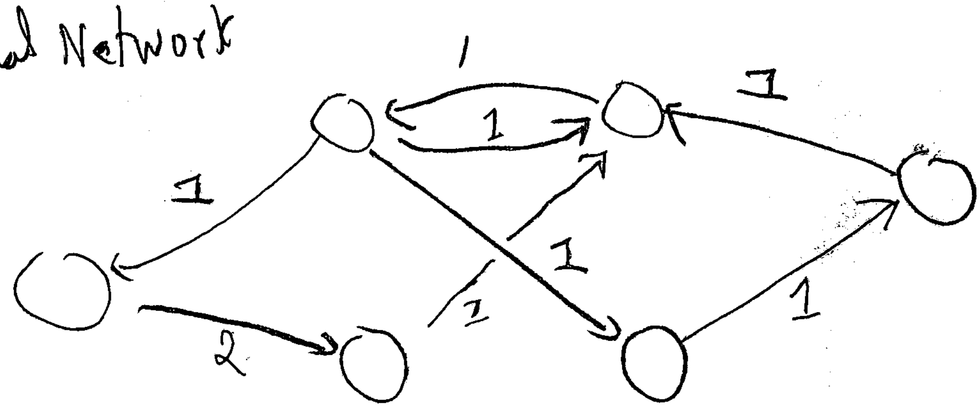
$$E_f = \{ (u,v) \in V^2 \mid c_f(u,v) > 0 \}$$

$$G_f(V, E_f)$$

Network & flow



Residual Network



Ford-Fulkerson

F-F Method (G, s, t, c)

1) Initialize flow f to 0
2) <u>While</u> $\exists$ aug path p in G_f

   Set $f = p + f$
3) return f

<u>Lemma</u> f flow on G, G_f the residual

a) f' is flow on G_f iff $f+f'$ flow on G.

b) f' is max-flow on G_f iff $f+f'$ is max flow on G

c) $|f+f'| = |f| + |f'|$ (f' has no flow into s)

<u>proof</u> a) $f'(e) \leq c_f(e)$ iff $f'(e) \leq c(e) - f(e)$

iff $(f'+f)e \leq c(e)$

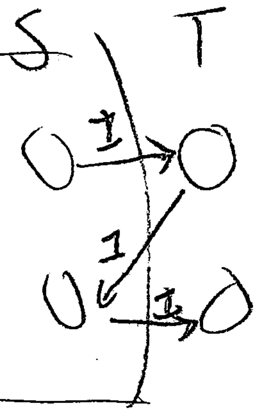
S, T cuts

Def S, T is a cut if

1) $S \cap T = \emptyset$ & $S \cup T = V$
2) $s \in S$ & $t \in T$

$$Cap(S,T) \equiv \sum_{u \in S,\, v \in T} c(u,v)$$

eg



Cap = 2

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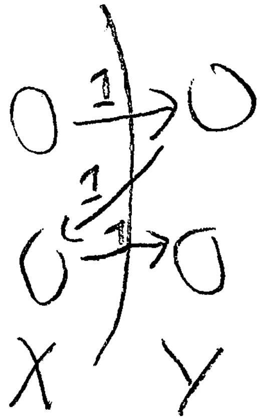
f flow

$$f(X,Y) \equiv \sum_{u \in X} \sum_{v \in Y} f(u,v)$$

eg



flow = 1

$f(S,T) \equiv$ net flow from S to T

$|f| \equiv f(s, V-s)$

<u>Lemma</u> f flow on G & (S,T) is a Cut

Then $|f| = f(S,T)$.

<u>proof</u> induction on $|S|$

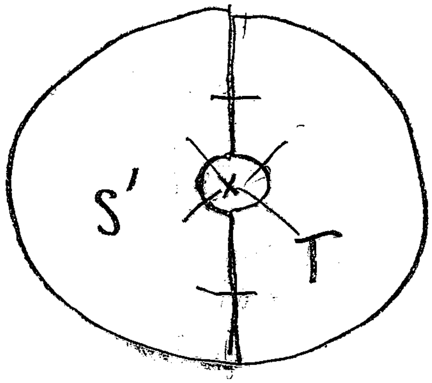
$|S| = 1$ done

assume true for cut (S',T') where $|S'|+1 = |S|$

$S = S' \cup \{x\}$ $\quad x \notin S'$ $\quad x \neq s$

$T = T' - \{x\}$

$T' = T \cup \{x\}$



$$f(S,T) = f(S',T) + f(x,T)$$

$$f(S',T') = f(S',T) + f(S',x)$$

$$f(S,T) - f(S',T') = f(x,T) - f(S',x) = f(x,T) + f(x,S')$$

$$= f(x, V - \{x\}) = 0$$

$$\therefore \; |f| \overset{\text{induct}}{=} f(S',T') = f(S,T)$$

Cor S,T cut $|f| \leq c(S,T)$

since $f(S,T) \leq c(S,T)$

Thm max-flow = min-cut

ie the following are equivalent

1) f is max-flow

2) G_f contains no augmenting paths.

3) $\exists$ cut S,T s.t. $|f| = cap(S,T)$

proof 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 1)

1) $\Rightarrow$ 2) i.e. $\neg$2) $\Rightarrow$ $\neg$1)

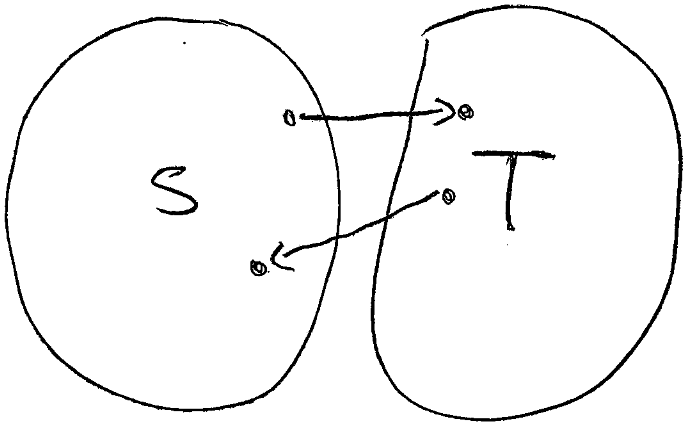
$\exists$ augmenting path $\Rightarrow$ f is not maximum

2) $\Rightarrow$ 3)

Let $S \subseteq V \equiv$ reachable vertices from s in G_f

$T = V - S$ note: $t \in T$ by 2).

In G



1) f saturates all edges from S to T

2) f does not use edges from T to S

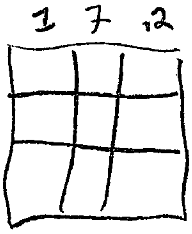
$\therefore |f| = f(S,T) = cap(S,T)$

3) $\Rightarrow$ 1) clear

An Image Segmentation Prob

Foreground / Background Prob

Input: 1) Pixel Image   eg



2) Affinity graph. $G=(V,E)$

weights $P_{ij} \equiv$ similarity of pixel V_i & V_j

3) $b_j \equiv$ likelihood V_j in background

$a_j \equiv$ " " foreground

Output: Partition A, B of V s.t.

$$\max_{A,B} g(A,B) = \sum_{i \in A} a_i + \sum_{i \in B} b_i - \sum_{\substack{i \in A \\ j \in B}} P_{ij} \quad (i,j) \in E$$

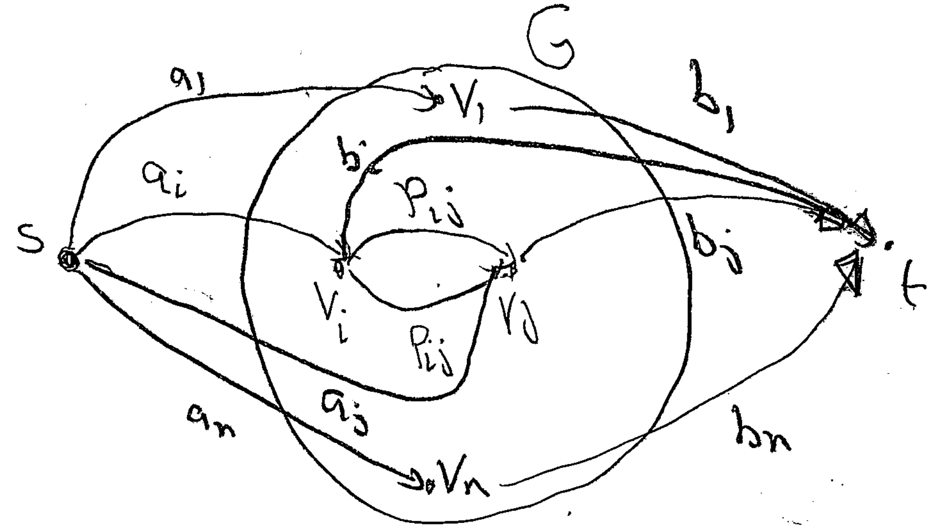
Change to Min Prob.

$Q \equiv \sum_i (a_i + b_i)$ then

$$g(A,B) = Q - \sum_{i \in A} b_i - \sum_{i \in B} a_i - \sum P_{ij}$$

Goal: $\min g'(A,B) = \sum_{i \in A} b_i + \sum_{i \in B} a_i + \sum P_{ij}$

Make a flow-Network:

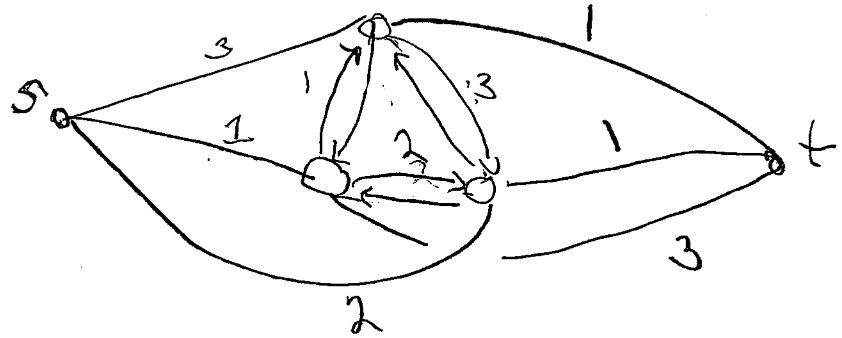


<u>Claim</u> If A, B is a cut then $s \in A$ & $t \in B$

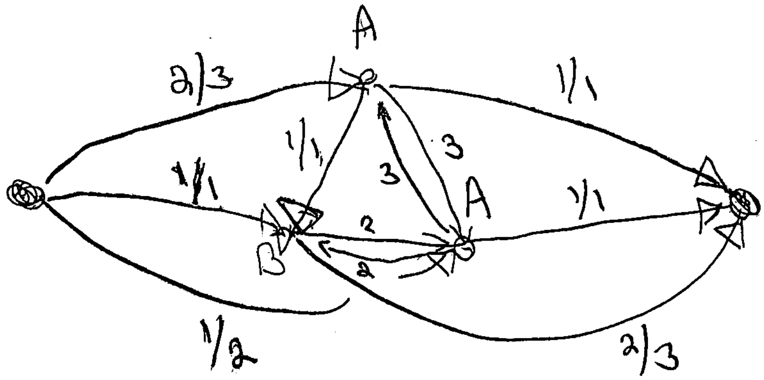
$$C(A,B) = q'(A,B).$$

<u>Thus</u> Min Cut is a Max Segmentation.

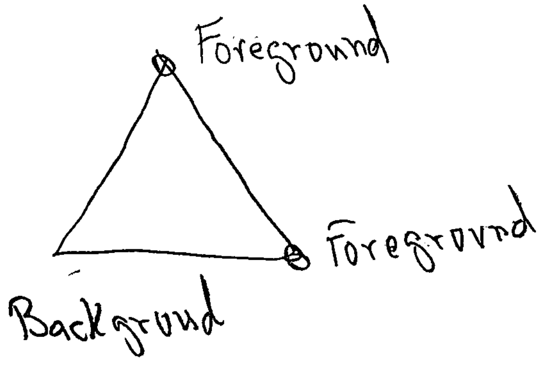
Examples



Max Flow



answer



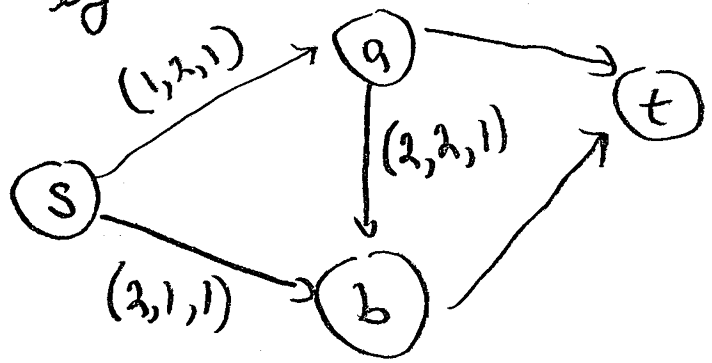
Dynamic Networks

Input: Network: $G=(V,E)$

Discreet time: $0 \le t \le 2$

Capacities: $C_{ij}^{t} \equiv$ cap from a_i to a_j at time t.

eg



Buffer size: b_a^t

Idea: Make 4 copies of static network and add edges between copies.