Parallel Algorithms III

Topics

1) Randomized List-Ranking
A linked list

Def Directed graph with head & tail
1) indegree = 1 except indegree(head) = 0
2) outdegree = 1 except outdeg(tail) = 0
3) Connected

Further Properties needed for parallel Alg.

1) Pointers in consecutive memory
2) Each pointer has an index which we can access in unit time.
**Random-Mate (Randomized Alg)**

1) **Contraction Phase**

1) Each live node randomly picks a sex.

2) If $F \xrightarrow{a} M \xrightarrow{b} X$, then

\[ \text{dies} \]

3) Stop when head points to nil.

(only head is live)
Thm: The contraction phase stops in \(c \log n\) rounds with high prob.

Let \(P_i\) = Event that node \(i\) is still live after one round.

Note: node \(i\) not head then \(\text{Prob}[P_i] = \frac{3}{4}\)

Let \(P_i^k\) = Event that node \(i\) still live after \(k\) rounds.

Note: \(\text{Prob}[P_i^k] = \left(\frac{3}{4}\right)^k\) if \(i\) not head. (By independence)

Set \(k = \frac{c \log_{4/3} n}{h}\)

\[
\text{Prob}[P_i^k] = \frac{1}{(4/3)^k} = \left(\frac{4}{3}\right)^{-c \log_{4/3} n} = \frac{1}{h^c}
\]
Let $P^k = \text{Event that some non-head node is still live.}$

Assume that node $0$ is the head.

$$P^k = P^k_1 \cup P^k_2 \cup \ldots \cup P^k_n$$

$$\text{Prob} \left[ P^k \right] = \text{Prob} \left[ P^k_1 \cup \ldots \cup P^k_n \right]$$

$$\leq \text{Prob} \left[ P^k_1 \right] + \ldots + \text{Prob} \left[ P^k_n \right]$$

$$\leq n \cdot \frac{1}{n^c} = \frac{1}{n^{c-1}}$$

If we set $c = 2$ then the contraction phase stops with prob $\leq \frac{1}{n}$ in $2 \log_3 n$ rounds.
In the expansion phase we run contraction phase "backwards".

\[ \text{live} \rightarrow \text{dead} \rightarrow \text{live} \]

\[ \text{dist} = c \]

\[ \text{live} \rightarrow \text{live} \rightarrow \text{live} \]

\[ \text{dist} = a + d \]

1) The same as "down" in Prescon.

Note: \( P \cdot T = O(n \log n) \) with high prob.

Goal: An Alg \( P \cdot T = O(n) \) with HP.
A Simple Randomized List-Ranking

Assume linked-list is doubly linked.

Alg Splicing-out
1) Make \( \frac{n}{\log n} \) queues of size \( \log n \) (\( \text{queue/proc} \))
2) Set sex of all nodes to \( M_0 \).
3) Reset sex of each queue-top to random sex.
4) If top is \( F \) and points to \( M \) then "splice-out" top.
5) Repeat while some queue not empty.

Thm After \( O(\log n) \) rounds all queues are empty with high probability.
Queue size $\log n$
Chernoff Bounds

Let $X_1, \ldots, X_t$ be independent 0/1 random variables.

Assume $\text{Prob}(X_i = 1) = p$

The binomial random variable is

$$S_n^p = X_1 + \ldots + X_t$$

$$\text{Expect}(S_t^p) = \sum E(X_i) = p \cdot t$$
\text{Theorem} \quad \text{Prob}(S^p_t < (1-\beta)pt) < e^{-\beta^2 pt/2} \quad \forall 0 \leq \beta < 1 \\
\text{Theorem} \quad \text{Prob}(S^p_t > (1+\beta)pt) < e^{-\beta^2 pt/2} \quad \forall 0 \leq \beta < 1
Let's fix one of the queues, say $Q$.

At a given round the prob Top is spiced-out is $\geq \frac{1}{4}$.

View prob as:

We have a coin $\text{Prob (Head)} = \frac{1}{4}$ $\text{Prob (Tail)} = \frac{3}{4}$

Question: After $t$ flips what is $\text{Prob}[\text{#heads} < \log n]$?

Suppose we pick $t$ s.t. $\text{Expect \#heads} = 4 \log n$.

ie $t = 16 \log n$

Here: $\log = \log$ base $2$. Could use base $e$.
We apply Chernoff with $p = \frac{1}{4}$, $t = 16 \log n$, $\beta = \frac{3}{4}$.

\[
\text{Prob}\left( S^p_t < (1-\beta)pt \right) < \exp\left(-p^2t/2\right)
\]

\[
\text{Prob}\left( S^p_t < \log n \right) < \exp\left(-\left(\frac{3}{4}\right)^2(\frac{3}{4}) 16 (\log n) \left(\frac{1}{2}\right)\right)
\]

\[
= \exp\left(-\frac{9}{8} \log n \right) \leq n^{-9/8}
\]

Thus, Prob that some queue is not empty after $t = 16 \log n$ rounds is $(n^{-9/8})^n \leq n^{-9/8}$. 

\[
\leq n^{-9/8}
\]