15-750

Graduate Algorithms
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Requirements

Grading

HW 30%
Midterm 30%
Final 40%

Discussion Piazza

Web page rglmiller

Grades A+ to B-

Audit ≥ B-
Course Goals

1) Understand many known:
   a) Algorithms
   b) Design Techniques
2) Analyze algorithm efficiency
3) Algorithm correctness
4) Communicate about code
5) Know key words
6) Design your own alg.
Machine Models

RAM = Random Access Machine

- Caching models
- Memory hierarchy
- Pipelining

Parallel Models

PRAM

Circuits
unit time ops
memory read/write
+,-,x,÷
Asymptotic Complexity

**Def:**

\[ f(n) \in \mathcal{O}(g(n)) \text{ if} \]

\[ \exists c, n_0 \geq 0 \forall n \geq n_0 \quad f(n) \leq cg(n) \]

\[ \mathcal{O}(g(n)) = \{ f(n) \mid f(n) \in \mathcal{O}(g) \} \]

\[ f \in o(g(n)) \text{ if} \]

\[ \forall c > 0 \exists n_0 \geq 0 \forall n \geq n_0 \quad f(n) \leq cg(n) \]

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1) \( f \in \mathcal{O}(g) \) if \( g \in \mathcal{O}(f) \)

2) \( f \in \mathcal{O}(g) \) if

\[ \exists c > 0 \forall n_0 \geq 0 \exists n_i = n_0 \quad f(n_i) \geq cg(n_i) \]

(infinitely often)
Claim \( 2n^2 + n + 1 \in O(n^2) \)

Try setting \( c = 3 \)

Solve \( 2n^2 + n + 1 = 3n^2 \)

Need \( n^2 - n - 1 \geq 0 \) OK for \( n_0 = 2 \)

\[ \forall n \geq 2 \quad (2n^2 + n + 1) \leq 3n^2 \]

L'Hôpital's Rule

\[
\lim_{{n \to \infty}} \frac{2n^2 + n + 1}{n^2} = 2, \quad \Rightarrow \quad c = 2 + \varepsilon \text{ works}
\]

Claim \( 3n \in o(n^2) \) \quad \text{set } n_0 = \frac{3}{c}

I need \( n_0 \) as \( \varepsilon \) is \( 0 \) from \( c \)

Solve \( 3n = Cn^2 \)

\[ 3 = cn \]

\[ n = \frac{3}{c} \]
Matrix Multiplication

Several Definitions

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\text{Real} \\
\text{Facts} 1 \quad (AB)C = A(BC) \\
2) A(B+C) = AB + AC \\
3) A \cdot B \neq B \cdot A \\
4) \lambda A = (\lambda \cdot 0) A \\
5) AB = C \Rightarrow C = \sum A_{xj} B_{ij} \quad \text{(outer products)}
\end{array}
\end{array}
\end{align*}
\]
Matrix Multiplication

Naive: $A, B$ are $n \times n$ matrices over Reals

$\text{Def } A \cdot B = C \text{ if } C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

$n^3$ multiplications \quad $(n-1)n^2$ additions

$O(n^3)$ operations

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Recursive Algorithm $M(A, B)$ \quad $n = 2^k$

1) if $A$ is $1 \times 1$ then return $a_{11} \cdot b_{11}$

2) Write $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ \quad $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$

$A_{i,j}$ are $\frac{n}{2} \times \frac{n}{2}$ \quad $B_{i,j}$ are $\frac{n}{2} \times \frac{n}{2}$

3) $C_{ij} = M(A_{i1}, B_{i1}) + M(A_{i2}, B_{i2})$

4) return $\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$
**Correctness**

Induction on $n$

$n = 1$ done

Assume $M(A, B) = A \cdot B$ for $n < n_0$

We know $C_{ij} = A_{ii} B_{jj} + A_{i2} B_{2j}$

Thus $C_{ij} = M(A_{i1}, B_{1j}) + M(A_{i2}, B_{2j})$

**Timing**

Let $T(n)$ = number of ops for $n \times n$

$T(n) \leq 8T(n/2) + cn^2$ & $T(1) = 1$

Claim: $T(n) = O(n^3)$

Consider recurrence

$T(n) = 7T(n/2) + cn^2$ & $T(1) = 1$

Claim: $T(n) = O(n^{\log_2 7})$
Solving Recurrences

Methods

1) Use formula,
2) Induction on n
3) Consider tree of recursive calls

3) \[ \text{Prob} \rightarrow \text{Work} \rightarrow cn^2 \]

\[ \frac{n}{2} \rightarrow 8c\left(\frac{n}{2}\right)^2 = 2cn^2 \]

\[ \left(\frac{3}{2}\right)^2 c\left(\frac{n}{2}\right)^2 = 2^3 cn^2 \]

1) \[ 2^\log n \cdot cn^2 \]

\[ O(n^3) \]
For 7 calls

\[ \sqrt{n} \]

\[ 7 \in (n^{3/2}) = \frac{7}{4} n^{3/2} \]

\[ 7^2 c(n^{1/2}) = (\frac{7}{4})^3 c n^2 \]

\[ (\frac{7}{4})^3 c n^2 = \frac{\sqrt{n^{10}}}{n^{1/4} c n^2} \]

\[ = c n^{\log_7} \]

\[ \text{Total } O(n^{\log_7}) \]
Strassen's Matrix Mult.

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
E & F \\
G & H
\end{bmatrix} =
\begin{bmatrix}
S_1 + S_2 & S_3 - S_5 \\
S_4 & S_6 + S_7
\end{bmatrix}
\]

\[
S_1 = (B - D)(G + H) \\
S_2 = (A + D)(E + H) \\
S_3 = (A - C)(E + F) \\
S_4 = (A + B) \cdot H \\
S_5 = A \cdot (F - H) \\
S_6 = D \cdot (G - E) \\
S_7 = (C + D) \cdot E
\]
Correctness

e.g. \( C \cdot E + D \cdot G = S_6 + S_7 \)

\[
S_6 + S_7 = D \cdot (G - E) + (G + D) \cdot E
\]

\[
= D \cdot G - D \cdot E + C \cdot E + D \cdot E
\]

\[
= D \cdot G + C \cdot E
\]
EXERCISES

6.1 Show that the integers modulo \( n \) form a ring. That is, \( \mathbb{Z}_n \) is the ring \( \{0, 1, \ldots, n - 1\} \), \(+, \cdot, 0, 1\), where \( a + b \) and \( a \cdot b \) are ordinary addition and multiplication modulo \( n \).

6.2 Show that \( M_n \), the set of \( n \times n \) matrices with elements chosen from some ring \( R \), itself forms a ring.

6.3 Give an example to show that the product of matrices is not commutative, even if the elements are chosen from a ring in which multiplication is commutative.

6.4 Use Strassen's algorithm to compute the product

\[
\begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix}
\begin{bmatrix}
5 & 6 \\
7 & 8
\end{bmatrix}.
\]

6.5 Another version of Strassen's algorithm uses the following identities to help compute the product of two \( 2 \times 2 \) matrices.

\[
\begin{align*}
s_1 &= a_{21} + a_{22} & m_1 &= s_3 s_8 & t_1 &= m_1 + m_2 \\
s_2 &= s_1 - a_{11} & m_2 &= a_{11} b_{11} & t_2 &= t_1 + m_4 \\
s_3 &= a_{11} - a_{21} & m_3 &= a_{12} b_{21} \\
s_4 &= a_{12} - s_2 & m_4 &= s_3 s_7 \\
s_5 &= b_{12} - b_{11} & m_5 &= s_1 s_8 \\
s_6 &= b_{22} - s_5 & m_6 &= s_4 b_{22} \\
s_7 &= b_{22} - b_{12} & m_7 &= a_{22} s_8 \\
s_8 &= s_6 - b_{21}
\end{align*}
\]

The elements of the product matrix are:

\[
\begin{align*}
c_{11} &= m_2 + m_3, \\
c_{12} &= t_1 + m_5 - m_6, \\
c_{21} &= t_2 - m_7, \\
c_{22} &= t_2 + m_4.
\end{align*}
\]

Show that these elements compute Eq. (6.1). Note that only 7 multiplications and 15 additions have been used.

\[\dagger\text{We can get around the detail that NUM}(a_i) \text{ is the integer representing the reverse of } a_i \text{ by taking the "jth row" of } B_i \text{ to be the } j \text{th row from the bottom instead of the top as we have previously done.}\]
What is a Space Efficient Strassen?

1) Add in place
2) Malloc $3n^2$ space per call.
3) Do find additions in output space of parent.

$W(n)$ be space used

\[ W(n) = 3n^2 + W(\frac{n}{2}) \]
\[ = 3n^2 + 3\left(\frac{n}{2}\right)^2 + 3\left(\frac{n}{4}\right)^2 + \cdots \]
\[ = 3n^2\left(1 + \frac{1}{4} + \frac{1}{16} + \cdots \right) \]

Note $1 + \alpha + \alpha^2 + \cdots = \frac{1}{1-\alpha}$, $\alpha < 1$

\[ = 3n^2\left(\frac{1}{1-\frac{1}{4}}\right) = 3n^2\left(\frac{4}{3}\right) = 4n^2 \]